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A study of a stably stratified hydromagnetic fluid in a rotating cylinder

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Slow, steady, mechanically driven, axisymmetric motion of a stably stratified, electrically conducting, rotating fluid is studied. Attention is focused upon the parameter values for which hydromagnetic effects first become important in a rotating stratified fluid and upon the nature of their influence on the interior flow of that fluid. It is found that hydromagnetic effects are able to alter the flow of a stratified rotating fluid at much weaker magnetic field strengths than the flow of an unstratified fluid. Specifically, the interior azimuthal flow is altered if $E/\alpha^2 \ll \sigma S \ll 1$ or if $E \ll \alpha^2$ and $1 \ll \sigma S$, where $E = \nu/\Omega L^2$, $\alpha^2 = \bar{\sigma} B^2/\rho\Omega$ and $S = \bar{\alpha}\Delta T g\nu/\Omega^2\kappa L$. The hydromagnetic effects act to decrease the vertical shear in the azimuthal flow from the levels which would occur in the absence of magnetic fields.

1. Introduction

In the past decade there has been a surge of interest in the study of rotating fluid flows including the effects of stratification and of magnetic fields. This has been motivated in part by a desire to understand better the planetary and solar interiors wherein rotating stratified hydromagnetic fluids are believed to occur. A great deal of work has been done on the behaviour of contained rotating fluids which are homogeneous (beginning with Greenspan & Howard 1963), stably stratified (beginning with Barcilon & Pedlosky 1967*a*, *b*) or hydromagnetic (beginning with Gilman & Benton 1968) but until recently no work has been done on the behaviour of a contained, rotating, stably stratified, hydromagnetic fluid.

In a closely related paper Loper (1975, herein referred to as L) analysed the steady flow of a stably stratified hydromagnetic fluid confined between two rotating infinite flat plates using a similarity transformation. He found that, with constant-heat-flux boundary conditions, the interior fluid exhibits columnar behaviour $[\partial v/\partial z < O(1)]$ if the magnetic interaction parameter α^2 is larger than $E^{\frac{1}{2}}$, regardless of the size of the stratification parameter σS. [For definitions of these parameters, see (2.6).] This is in contrast to non-magnetic flows, where laminated flow $[\partial v/\partial z = O(1)]$ occurs if $E^{\frac{1}{2}} \ll \sigma S$, and in contrast to unstratified flows, where magnetic effects are important only if $1 \ll \alpha^2$. Since similarity solutions may yield solutions which do not model the flow of completely contained fluids, the present paper will investigate whether the results of L are valid for contained fluids.

Typically, when one wishes to analyse the behaviour of a fluid under new conditions, a relatively simple problem is formulated which retains the physics but which poses a minimum of mathematical complications. The present study is no exception. In this paper we shall consider the nature of the flow of an electrically conducting fluid which fills a rotating right circular cylinder. In particular the magnitude of the vertical shear $\partial v/\partial z$ of the zonal velocity is studied. The fluid has a thermally imposed stable density gradient and we specify constant-heat-flux boundary conditions. The cylinder is assumed to be an electrical insulator and a uniform axial magnetic field is applied. The flow is mechanically driven by differential motion of the upper and lower boundaries. This problem may be thought of as an extension of that studied by Barcilon & Pedlosky (1967*a*, *b*) to include magnetic effects or as an extension of that studied by Ingham (1969) and Vempaty & Loper (1975) to include stratification.

The simultaneous action of Coriolis, buoyant and magnetic forces is certain to exhibit a rich variety of new phenomena and to yield new length scales and force balances. Therefore a survey of the possible length scales and force balances which may occur is presented in appendix A after the problem has been formulated mathematically in §2. Following this, the flow problem is solved in §§3–6. The interior is analysed in §3, the boundary layers on the end walls are analysed in §4, the side-wall boundary layers are studied in §5 and the problem is closed in §6. Finally a summary and a discussion of the results are presented in §7.

2. Mathematical formulation

We shall take as our starting point equations L(2.10)–L(2.17) [i.e. equations (2.10)–(2.17) of Loper 1975] with the Froude number set equal to zero; in this paper we shall concentrate upon the response of the fluid to only the mechanical forcing and shall neglect the thermally driven flow studied in L.†

The equations may be simplified by simultaneous integration of L(2.14) and L(2.16) to yield

$$\psi_z + (\nabla^2 - r^{-2})\,\phi = 0.$$

This allows L(2.10) to be written as

$$2v = p_r + 2\alpha^2\psi_z - E(\nabla^2 - r^{-2})\,\psi_z \tag{2.1}$$

and the variable ϕ has been eliminated from direct consideration. We may also eliminate the pressure from direct consideration by cross-differentiation of L(2.12) and (2.1) to arrive at the set of equations

$$\underset{[1]}{2v_z} = \underset{[2]}{T_r} + \underset{[3]}{2\alpha^2\psi_{zz}} - \underset{[4]}{E(\nabla^2 - r^{-2})^2\psi}, \quad \underset{[5]}{2\psi_z} = \underset{[6]}{2\alpha^2 b_z} + \underset{[7]}{E(\nabla^2 - r^{-2})v}, \tag{2.2), (2.3}$$

$$\underset{[8]}{-\sigma S r^{-1}(r\psi)_r} = \underset{[9]}{E\nabla^2 T}, \quad \underset{[10]}{v_z} + \underset{[11]}{(\nabla^2 - r^{-2})\,b} = 0, \tag{2.4), (2.5}$$

† The assumption that $\epsilon \ll 1$ in L(2.9) may be insufficient to ensure the validity of the present analysis. A more severe restriction on ϵ may be necessary to maintain linearity in the side-wall boundary layers (see Barcilon 1970).

where $\nabla^2 = \partial^2/\partial r^2 + r^{-1}\,\partial/\partial r + \partial^2/\partial z^2$ and

$$E = \nu/\Omega L^2, \quad \sigma S = \overline{\alpha}\Delta T g\nu/\kappa\Omega^2 L, \quad 2\alpha^2 = \overline{\sigma}B^2/\rho_0\,\Omega. \tag{2.6}$$

The variables governed by (2.2)–(2.5) are the azimuthal velocity v, the fluid stream function ψ, the perturbation temperature T and the azimuthal magnetic field (or electric current stream function) b while $\Omega, L, \nu, \overline{\alpha}, \Delta T, g, \kappa, \overline{\sigma}, B$ and ρ_0 are respectively the rotation rate, cylinder height, kinematic viscosity, thermal expansion coefficient, temperature difference, acceleration due to gravity, thermal diffusivity, electrical conductivity, axial magnetic field strength and mean density. The terms in (2.2)–(2.5) have been numbered to allow individual reference to them. Four types of forces have been included in the momentum equations (2.2) and (2.3): viscous ([4], [7]), Coriolis ([1], [5]), buoyancy ([2]) and magnetic ([3], [6]).

The boundary conditions for the problem are

$$\psi = \psi_z = T_z = b = 0, \quad v = v_B(r) \quad \text{at} \quad z = 0, \tag{2.7a}$$

$$\psi = \psi_z = T_z = b = 0, \quad v = v_T(r) \quad \text{at} \quad z = 1, \tag{2.7b}$$

$$\psi = \psi_r = T_r = b = v = 0 \quad \text{at} \quad r = r_0, \tag{2.7c}$$

where Lr_0 is the radius of the cylinder.

As a guide to the subsequent analysis, a survey of the possible length scales inherent in (2.2)–(2.5) is presented in appendix A. The results are given in tables 1–3 and in figures 1 and 2. The co-ordinate axes in figures 1 and 2 are defined by

$$\alpha^2 = E^{-x}, \quad \sigma S = E^{-y}. \tag{2.8}$$

It may be seen from figure 2 that the Ekman scale $A1$ exists provided that

$$\sigma S \ll E^{-1}, \quad \alpha^2 \ll 1. \tag{2.9}$$

If (2.9) is satisfied, we may replace conditions (2.7) with the Ekman–Hartmann compatibility conditions simplified for $\alpha^2 \ll 1$ (see Loper 1975):

$$\left.\begin{aligned} \psi = \tfrac{1}{2}E^{\frac{1}{2}}\,[v_B(r)-v], \quad b = -\tfrac{1}{2}E^{\frac{1}{2}}\,[v_B(r)-v] \\ T_z = \sigma S[rv_B(r)-rv]_r/2r \end{aligned}\right\} \quad \text{at} \quad z = 0, \qquad \begin{aligned}(2.10a, b)\\(2.10c)\end{aligned}$$

$$\left.\begin{aligned} \psi = -\tfrac{1}{2}E^{\frac{1}{2}}\,[v_T(r)-v], \quad b = \tfrac{1}{2}E^{\frac{1}{2}}\,[v_T(r)-v] \\ T_z = \sigma S[rv_T(r)-rv]_r/2r \end{aligned}\right\} \quad \text{at} \quad z = 1, \qquad \begin{aligned}(2.10d, e)\\(2.10f)\end{aligned}$$

$$\psi = \psi_r = T_r = b = v = 0 \quad \text{at} \quad r = r_0. \tag{2.10g}$$

When conditions (2.10) are employed, the axial viscous terms of (2.2) may be consistently neglected. This completes the formulation of the problem and we now turn to investigation of the nature of the interior flow.

Scale number	Balance terms in (A 5) and (A 7)	Important forces: Coriolis	Meridional viscous	Azimuthal viscous	Buoyant	Magnetic	Dominant terms in equations: (2.2)	(2.3)	(2.4)	(2.5)
1	*a*, *c*	×	×	×	·	·	1, 4	5, 7	—	—
2*a*	*a*, *b*	·	×	×	·	×	3, 4	6, 7	—	10, 11
2*b*	*d*, *e*	·	·	×	·	×	—	6, 7	—	10, 11
3	*b*, *c*	×	·	·	·	×	1, 3	5, 6	—	10, 11
4	*a*, *d*	·	×	·	×	·	2, 4	—	8, 9	—
5	*c*, *d*	×	·	×	×	·	1, 2	5, 7	8, 9	—
6	*b*, *e*	·	·	·	×	×	2, 3	—	8, 9	—
7	*c*, *e*	×	·	·	×	×	1, 2	5, 6	8, 9	10, 11

TABLE 1. Important forces and dominant terms for various balances for both vertical scales (case *A*) and horizontal scales (case *B*)

Scale number	Balance terms in (A 5)	Scale name	Dimensionless vertical scale	Dimensional vertical scale	Order in z	Ordinary or partial
1	a, c	Ekman	$E^{\frac{1}{2}}$	$(\nu/\Omega)^{\frac{1}{2}}$	4	O
2a	a, b	Hartmann	$(E/\alpha^2)^{\frac{1}{2}}$	$(\rho\nu/\bar{\sigma}B^2)^{\frac{1}{2}}$	4	O
2b	d, e	Hartmann	$(E/\alpha^2)^{\frac{1}{2}}$	$(\rho\nu/\bar{\sigma}B^2)^{\frac{1}{2}}$	2	O
3	b, c	—	—	—	—	—
4	a, d	Buoyant Ekman	$(l^2E^2/\sigma S)^{\frac{1}{6}}$	$(l_*^2\,\nu\kappa/\beta g)^{\frac{1}{6}}$	6	P
5	c, d	.	$(l^2/\sigma S)^{\frac{1}{2}}$	$(l_*^2\,\Omega^2\kappa/\beta g\nu)^{\frac{1}{2}}$	2	P
6	b, e	.	$(l^2E\alpha^2/\sigma S)^{\frac{1}{4}}$	$(l_*^2\bar{\sigma}B^2\kappa/\rho\beta g)^{\frac{1}{4}}$	4	P
7	c, e	.	$(l^2E/\sigma S\alpha^2)^{\frac{1}{4}}$	$(l_*^2\,\Omega^2\rho\kappa/\bar{\sigma}B^2\beta g)^{\frac{1}{4}}$	4	P

TABLE 2. Summary of vertical length scales (case A). In this table and table 3 an asterisk subscript denotes a dimensional length and $\beta \equiv \bar{\alpha}\Delta T/L$. Scales $A2b$ and $A7$ each control the interior for certain parameter values (see figure 3)

Scale number	Balance terms in (A 7)	Scale name	Dimensionless horizontal scale	Dimensional horizontal scale	Order in r	Ordinary or partial
1	a, c	Stewartson	$(hE)^{\frac{1}{3}}$	$(h_*\,\nu/\Omega)^{\frac{1}{3}}$	6	P
2a	a, b	Hydromagnetic	$(h^2E/\alpha^2)^{\frac{1}{4}}$	$(h_*^2\,\nu\rho/\bar{\sigma}B^2)^{\frac{1}{4}}$	8	P
2b	d, e	Hydromagnetic	$(h^2E/\alpha^2)^{\frac{1}{4}}$	$(h_*^2\,\nu\rho/\bar{\sigma}B^2)^{\frac{1}{4}}$	4	P
3	b, c	Vempaty	(h/α^2)	$h_*\rho\Omega/\bar{\sigma}B^2$	2	P
4	a, d	Buoyancy	$(E^2/\sigma S)^{\frac{1}{4}}$	$(\nu\kappa/\beta g)^{\frac{1}{4}}$	4	O
5	c, d	Hydrostatic	$(h^2\sigma S)^{\frac{1}{2}}$	$(h_*^2\,\beta g\nu/\Omega^2\kappa)^{\frac{1}{2}}$	2	P
6	b, e	—	—	—	—	—
7	c, e		$(E/\alpha^2\sigma S)^{\frac{1}{2}}$	$(\rho\Omega^2\kappa/\bar{\sigma}B^2\beta g)^{\frac{1}{2}}$	2	O

TABLE 3. Summary of horizontal length scales (case B). The $E^{\frac{1}{4}}$ side-wall layer, labelled $B8$ in the text, cannot be found by interior scale analysis

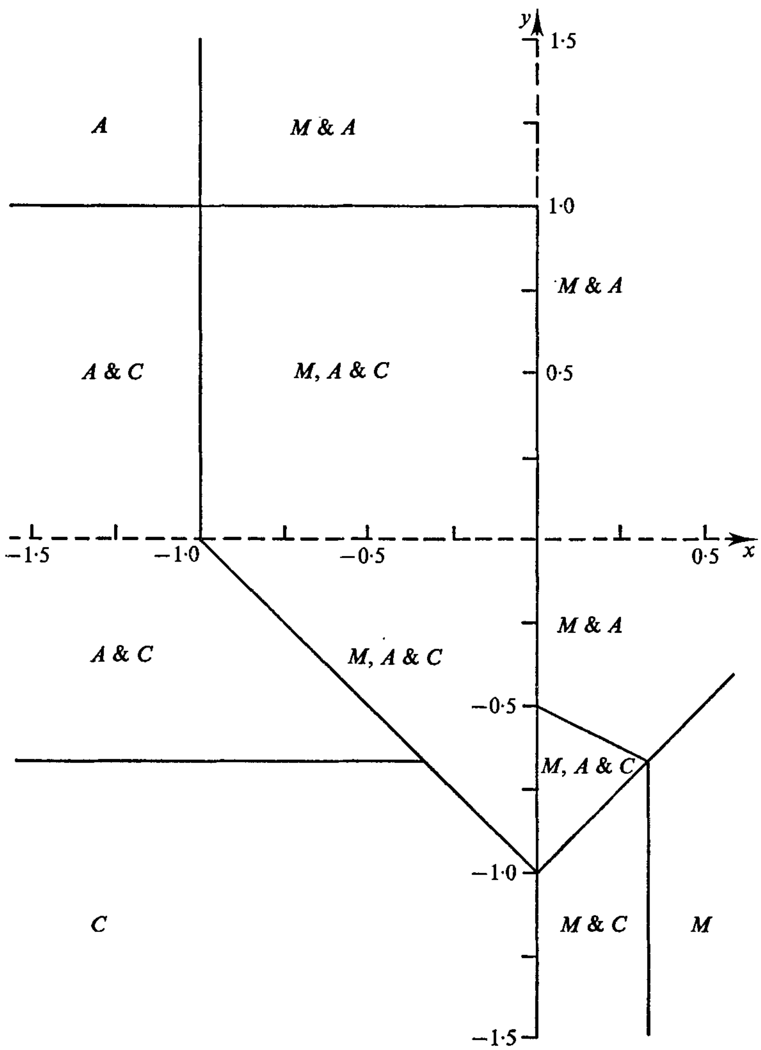

FIGURE 1. The x, y plane, showing the different regions in which buoyant (A), Coriolis (C) and magnetic (M) forces are important. Viscous forces are important on some scale in all regions. The parameters x and y are defined as $x = -\ln(\alpha^2/E)$, $y = -\ln(\sigma S/E)$.

3. The nature of the interior flow

In this section the qualitative nature of the interior flow is investigated for the parameter range $E \ll \sigma S \ll E^{-1}, \alpha^2 \ll 1$ [i.e. $-1 < y < 1, x < 0$, where x and y are defined by (2.8)], which contains most cases of interest. It is seen that the interior flow is controlled by the boundary layers which occur on the cylinder's end walls (at $z = 0, 1$) and appears to be independent of the side-wall boundary layers which occur at $r = r_0$. Detailed solutions for the end-wall boundary layers and interior flow are presented in § 4 while the solutions for the side-wall boundary layers are presented and discussed in § 5. The separate solutions found in §§ 4 and 5 are combined in § 6, verifying the simple picture presented in this section.

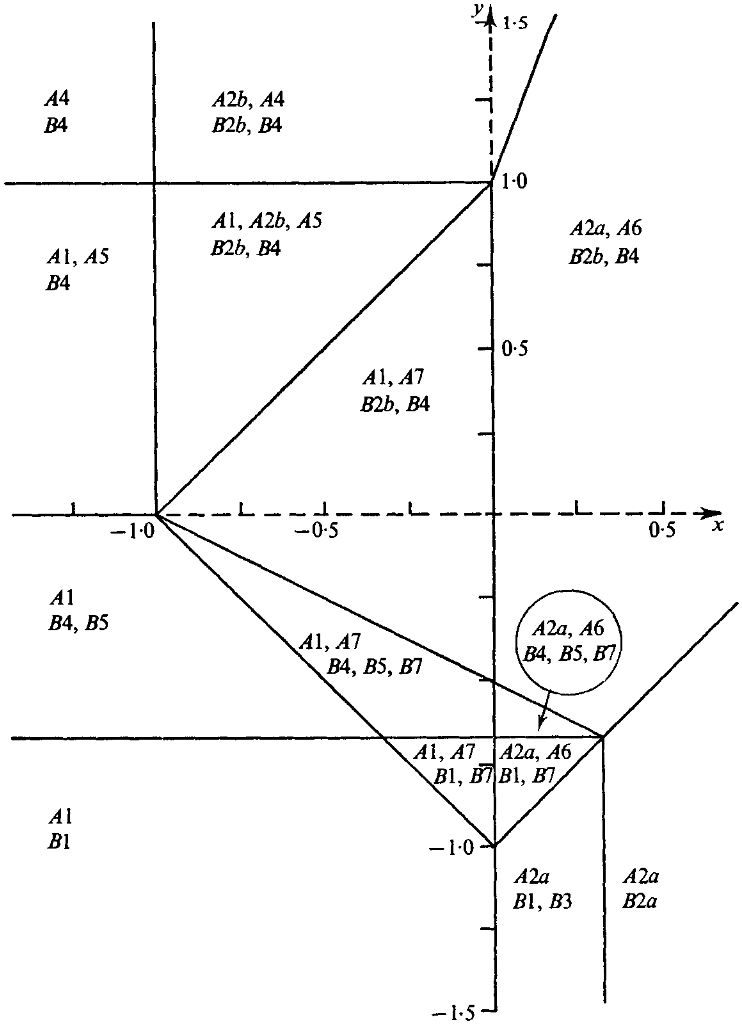

FIGURE 2. The x, y plane, listing the vertical (labelled A) and horizontal (labelled B) scales which occur for various parameter values. The nature of these scales is explained in tables 2–4.

It is known from Ingham (1969) and Vempaty & Loper (1975) that the interior azimuthal velocity of a homogeneous hydromagnetic fluid is the average of that of the end walls:

$$v^i = \tfrac{1}{2}[v_B(r)+v_T(r)], \tag{3.1}$$

for all values of α^2. In this case v_z^i is small. On the other hand Barcilon & Pedlosky (1967 *a*, *b*) found that in the absence of hydromagnetic effects

$$v_z^i = \begin{cases} O(\sigma S E^{-\frac{1}{2}}) & \text{if} \quad \sigma S \ll E^{\frac{1}{2}}, \\ O(1) & \text{if} \quad E^{\frac{1}{2}} \ll \sigma S. \end{cases} \tag{3.2}$$

The goal of this section is to determine the order of magnitude of v_z^i as a function of σS and α^2 when both buoyant and magnetic forces are present. It will be seen that the flow is affected by hydromagnetic forces for surprisingly small values of α^2 when σS is large.

The forcing for the problem is provided by the functions $v_B(r)$ and $v_T(r)$ in the boundary conditions (2.10). We shall begin the analysis of this section by assuming a laminated flow of arbitrary strength,

$$v_z^i = O(E^k), \tag{3.3}$$

then proceed to investigate the bounds of the size of the laminated flow which are necessary to maintain these forcing terms of dominant order.

The first step is to determine the orders of magnitude of the interior variables ψ^i, T^i and b^i using (2.2)–(2.5). We shall assume that in the interior $\partial/\partial r, \partial/\partial z = O(1)$ the viscous terms are small and term [3] of (2.2) is small. This last assumption may be verified *a posteriori*. From (2.2), (2.4) and (2.5) it is easily found that

$$(\psi^i, T^i, b^i) = O(E^{1+k}/\sigma S, E^k, E^k). \tag{3.4}$$

The dependence of the interior variables on the axial co-ordinate z is governed by (2.3), which yields

$$\psi_z^i = 0 + O(\alpha^2 E^k, E) \quad \text{if} \quad \sigma S\alpha^2 \ll E, \tag{3.5a}$$

or

$$b_z^i = 0 + O(E^{1+k}/\sigma S\alpha^2) \quad \text{if} \quad E \ll \sigma S\alpha^2. \tag{3.5b}$$

Because the viscous terms have been neglected in deriving (3.5), (3.5*a*) is valid only if $\sigma S \ll 1$. However (3.5*b*) is valid for all σS provided $E \ll \alpha^2$.

We may rewrite (2.4) as

$$T_{zz} = -r^{-1}(\sigma S E^{-1} r\psi + rT_r)_r. \tag{2.4a}$$

In any thin side-wall layer (2.4*a*) reduces to

$$\sigma S E^{-1}\psi + T_r = 0. \tag{3.6}$$

If we were to specify side-wall conditions at $r = r_0$ such that $\sigma S E^{-1}\psi + T_r \neq 0$, this would force $T_{zz}^i \neq 0$ since the side-wall layers cannot satisfy the boundary condition. However since we specify $\sigma S E^{-1}\psi + T_r = 0$, $T_{zz}^i = 0$ and (3.6) is valid both in the interior and in any side-wall layers. If (3.6) is valid then either (3.5*a*) or (3.5*b*) leads to the conclusion that the variables v_z^i, ψ^i and T^i are independent of z to dominant order.

Assuming that

$$E^{\frac{1}{2}} \ll v_z^i \quad (k < \tfrac{1}{2}), \tag{3.7}$$

(3.4) reveals that $E^{\frac{1}{2}} \ll b^i$. This makes the boundary condition on b in (2.10) homogeneous; the forcing terms $E^{\frac{1}{2}}v_B$ and $E^{\frac{1}{2}}v_T$ are small. To simplify the analysis we shall further assume that the forcing provided through the conditions on T_z is always negligible compared with the forcing for ψ in (2.10). This assumption may be verified *a posteriori*. In order that the conditions on ψ remain inhomogeneous, we must specify

$$v \leqslant O(1), \quad \psi \leqslant O(E^{\frac{1}{2}}) \tag{3.8), (3.9}$$

in all regions: interior and boundary layers on $z = 0$ and $z = 1$.

Using conditions (3.8) and (3.9) on the interior variables leads to the constraints

$$v_z^i \leqslant O(1) \qquad (k \geqslant 0), \tag{3.10}$$

$$v_z^i \leqslant O(\sigma S E^{-\frac{1}{2}}) \quad (k \geqslant -y \sim \tfrac{1}{2}). \tag{3.11}$$

These constraints are sufficient to reproduce the results of Barcilon & Pedlosky (1967*b*) given in (3.2).

Constraints (3.10) and (3.11) are valid for the entire parameter range under consideration ($E \ll \sigma S \ll E^{-1}$, $\alpha^2 \ll 1$). We shall now determine the additional constraints which arise owing to (3.8) and (3.9) being applied within the boundary layers on $z = 0, 1$.

As a result of assumption (3.7), the boundary conditions on b are homogeneous. This forces a correction term of the same size as b^i:

$$\tilde{b} = O(E^k),$$

where the tilde indicates a correction or boundary-layer variable. If $\sigma S\alpha^2 \ll E$, $\tilde{b}$ satisfies the equation

$$(\nabla^2 - r^{-2})\tilde{b} = 0,$$

which has no intrinsic scale. In this case no new constraints on v_z^i occur. However if $E \ll \sigma S\alpha^2$, $\tilde{b}$ satisfies boundary-layer equations near $z = 0$ and $z = 1$ of scale $A7$ if $\sigma SE \ll \alpha^2$ or of scale $A2b$ if $\alpha^2 \ll \sigma SE$. In either case additional constraints on v_z^i occur, as we shall now see.

Scale $A7$ exists if $E/\alpha^2 \ll \sigma S \ll \alpha^2/E$ and $\alpha^2 \ll 1$ $(-1-x < y < 1+x$ and $x < 0)$. In this case we see from table 2 that

$$\partial/\partial z = O(\sigma S\alpha^2/E)^{\frac{1}{4}} = O(E^{-\frac{1}{4}(1+x+y)}).$$

With $\tilde{b}_{A7} = O(E^{+k})$,

$$\tilde{v}_{A7} = O(E^k(\sigma S\alpha^2/E)^{\frac{1}{4}}) = O(E^{k-\frac{1}{4}(1+x+y)})$$

and

$$\tilde{\psi}_{A7} = O(E^k\alpha^2) = O(E^{k-x}).$$

From (3.8) and (3.9) it follows that

$$v_z^i \leqslant O(E/\sigma S\alpha^2)^{\frac{1}{4}} \quad (k \geqslant \tfrac{1}{4}(1+x+y)) \tag{3.12}$$

and

$$v_z^i \leqslant O(E^{\frac{1}{2}}/\alpha^2) \qquad (k \geqslant \tfrac{1}{2}+x). \tag{3.13}$$

Constraint (3.12) is more severe if $E/\alpha^2 \ll \sigma S \ll \alpha^2/E$ and $\alpha^6/E \ll \sigma S$ $(3x+1 < y)$ and $-1-x < y < 1+x)$ while constraint (3.13) is more severe if $E/\alpha^2 \ll \sigma S \ll \alpha^6/E$ and $\alpha^2 \ll 1(-1-x < y < 3x+1$ and $x < 0)$.

Scale $A2b$ exists if $\alpha^2 \ll \sigma SE \ll 1$ and $E \ll \alpha^2$ $(1+x < y < 1$ and $-1 < x)$. In this case we see from table 2 that

$$\partial/\partial z = O(\alpha^2/E)^{\frac{1}{2}} = O(E^{-\frac{1}{2}(1+x)}).$$

[It should be remarked that scale $A2b$ is the Hartmann scale, which is larger than the Ekman scale provided $\alpha^2 \ll 1$, and we may consistently use the Ekman–Hartmann compatibility conditions (2.10).] It is found that

$$\tilde{v}_{A2b} = O(E^k(\alpha^2/E)^{\frac{1}{2}}) = O(E^{k-\frac{1}{2}(1+x)})$$

and

$$\tilde{\psi}_{A2b} = O(E^k\alpha^2) = O(E^{k-x}).$$

Constraints (3.8) and (3.9) now lead to

$$v_z^i \leqslant O(E^{\frac{1}{2}}/\alpha) \quad (k \geqslant \tfrac{1}{2}+\tfrac{1}{2}x) \tag{3.14}$$

and (3.13) again. Throughout the region of existence of scale $A2b$, constraint (3.14) is more severe than (3.13).

In summary there are five constraints on the size of the laminated flow: two, (3.10) and (3.11), valid for all x and y and three, (3.12)–(3.14), valid for limited

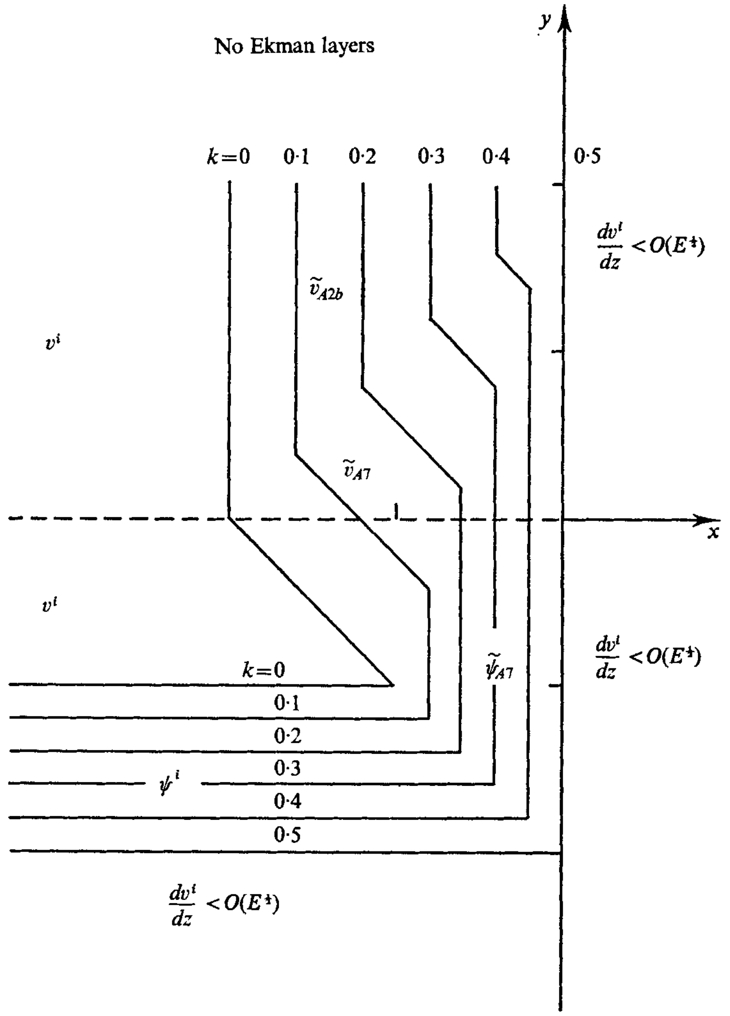

FIGURE 3. The strength of the laminated flow as a function of x and y. The parameter k is defined by $dv^i/dz = O(E^k)$. If $x > 0$ or $y < -1{\cdot}0$, dv^i/dz is less than $E^{\frac{1}{2}}$ and this theory does not apply. The symbols v^i, ψ^i, $\tilde{\psi}_{A7}$, $\tilde{v}_{A7}$ and $\tilde{v}_{A2b}$ denote the variable and scale which control the interior in each of five parameter regions.

ranges of x and y. Each of these constraints is dominant in a region of the x, y plane and controls the size of the lamination parameter k for values of the parameters lying within that region. The size of k for all five regions is contoured on the x, y plane in figure 3. The limiting variable for each region is noted on the figure.

For values of the parameters lying within the region of the x, y plane where $k = 0$ in figure 3, the interior flow is laminated to dominant order. In this case the Ekman layers are absent, the interior flow directly satisfies the boundary conditions on the azimuthal velocity at the end walls and meridional circulation is suppressed. A value of k less than zero on figure 3 indicates that laminated flow is of smaller than unit order; that is, v^i is independent of z to dominant order. In this case one would anticipate that the interior azimuthal velocity is the average of the top and bottom speeds: $v^i = \frac{1}{2}[v_B(r)+v_T(r)]$. It is seen in §4 that this is indeed the case.

The Ekman layers are absent to dominant order for parameter values lying within the regions of the x, y plane marked $\tilde{v}_{A7}$ and $\tilde{v}_{A2b}$ in figure 3. In these cases the interior azimuthal velocity is matched by the azimuthal velocity in the layers with scale $A7$ or $A2b$, and the order of the meridional circulation is smaller than $E^{\frac{1}{2}}$. It should be remarked that, for parameter values in region v_{A7}, an increase in the strength of the stratification leads to a *decrease* in the strength of the laminated flow.

The central assumption of this section is (3.7). It is seen from figure 3 that this assumption is valid if $E \ll \sigma S \ll E^{-1}$ and $\alpha^2 \ll 1$. If $\sigma S \ll E$ or $1 \ll \alpha^2$, the strength of the laminated flow is less than order $E^{\frac{1}{2}}$. It may be verified that for each of the five parameter regions in figure 3 the thermal boundary condition is homogeneous to dominant order. This completes the qualitative analysis of the interior flow; the detailed solutions for the interior flow and end-wall boundary layers are found in the next section.

4. Solutions for interior and end-wall boundary layers

Solutions of (2.2)–(2.5) subject to conditions (2.10) may now be found systematically for the various regions drawn on figures 2 and 3, relying upon the information obtained in §3. The analysis of this section and that for the side-wall layers in §5 will be completed for the parameter range $E \ll \sigma S \ll E^{-1}$, $\alpha^2 \ll 1$ [$-1 < y < 1$, $x < 0$; see (2.8)]. Our primary guide for this section is figure 3 and we shall consider the five regions marked ψ^i, v^i, $\tilde{v}_{A2b}$, $\tilde{v}_{A7}$ and $\tilde{\psi}_{A7}$ in turn. In what follows the dependent variables will be expanded in asymptotic series, the series being carried far enough to indicate the nature of the expansion. Part of this work duplicates that of Barcilon & Pedlosky (1967*b*) to dominant order but it is instructive to investigate the lower-order terms and see how they progressively become of dominant order.

4.1. *Solutions for parameter region* ψ^i

The fluid flow for this parameter region, $E \ll \sigma S \ll E^{\frac{1}{2}}$, $\sigma S\alpha^2 \ll E$ ($-1 < y < -\frac{1}{2}$, $y < -1-x$), is homogeneous and non-magnetic to dominant order. The stream function in the interior determines the strength of the laminated flow. There are no boundary layers on $z = 0$, 1 other than Ekman layers.

Relying on (3.5*a*) we may assume

$$v = v^{i0}(r) + \sigma S E^{-\frac{1}{2}} v^{i1}(r,z) + (\sigma S\alpha)^2 E^{-\frac{3}{2}} v^{i2}(r,z), \tag{4.1a}$$

$$\psi = E^{\frac{1}{2}}\psi^{i0}(r) + \sigma S \psi^{i1}(r,z) + \sigma S\alpha^2 E^{-\frac{1}{2}}\psi^{i2}(r,z), \tag{4.1b}$$

$$T = \sigma S E^{-\frac{1}{2}} T^{i1}(r,z) + (\sigma S\alpha)^2 E^{-\frac{3}{2}} T^{i2}(r,z), \tag{4.1c}$$

$$b = \sigma S E^{-\frac{1}{2}} b^{i1}(r,z) + (\sigma S\alpha)^2 E^{-\frac{1}{2}} b^{i2}(r,z). \tag{4.1d}$$

The side-wall boundary-layer variables have been omitted from (4.1) and from similar formulations later in this section. These variables are considered in §5. In expansion (4.1) variables with superscript $i1$ become important as $\sigma S \to E^{\frac{1}{2}}$, leading to the solutions presented in §4.2. Also variables with superscript $i2$ become important as $\sigma S\alpha^2 \to E$, leading to the solutions presented in §4.5.

Substitution of (4.1) into (2.2)–(2.5) yields

$$\nabla^2 T^{i1} = -(r\psi^{i0})_r/r, \quad 2v_z^{i1} = T_r^{i1}, \tag{4.2a,b}$$

$$(\nabla^2 - r^{-2})b^{i1} = -v_z^{i1}, \quad \psi_z^{i1} = 0; \tag{4.2c,d}$$

$$\psi_z^{i2} = b_z^{i1}, \quad \nabla^2 T^{i2} = -(r\psi^{i2})_r/r, \tag{4.2e,f}$$

$$2v_z^{i2} = T_r^{i2}, \quad (\nabla^2 - r^{-2})b^{i2} = -v_z^{i2}. \tag{4.2g,h}$$

The Ekman compatibility conditions (2.10) become

$$\psi^{i0} = \binom{+}{-}\frac{1}{2}\left[v^{i0} - \binom{v_T}{v_B}\right], \quad \psi^{i1} = \binom{+}{-}\tfrac{1}{2}v^{i1},$$

$$\psi^{i2} = b^{i1} = b^{i2} = T_z^{i1} = T_z^{i2} = 0 \quad \text{at} \quad z = \binom{1}{0}.$$

Since ψ^{i0} and v^{i0} are independent of z, we obtain

$$v^{i0} = \tfrac{1}{2}[v_B(r) + v_T(r)], \quad \psi^{i0} = \tfrac{1}{4}[v_B(r) - v_T(r)]. \tag{4.3}$$

Also $\quad v^{i1} = \tfrac{1}{16}[v_T(r) - v_B(r)](2z-1), \quad 8\psi^{i1} = T_r^{i1} = \tfrac{1}{4}[v_T(r) - v_B(r)],$

which are known solutions, while

$$b^{i1} = \int_0^r \frac{r^2-\eta^2}{16r}[v_B(\eta) - v_T(\eta)]\,d\eta$$
$$-\left\{\int_0^{r_0} \frac{r_0^2-\eta^2}{16r_0}[v_B(\eta) - v_T(\eta)]\,d\eta\right\} \sum_{m=0}^{\infty} \frac{4}{\pi(2m+1)} \frac{J_1[(2m+1)\pi r]}{J_1[(2m+1)\pi r_0]} \sin(2m+1)\pi z, \tag{4.4}$$

where we have satisfied the condition $b^{i1} = 0$ at $r = r_0$. The solutions for variables with superscript $i2$ are of less interest and will not be presented.

4.2. *Solutions for parameter region v^i*

In this parameter region, with $E^{\frac{1}{2}} \ll \sigma S$, the fluid flow is laminated to dominant order and the Ekman layers are absent. The expansion of the variables in this parameter region takes on two separate forms according as $\sigma S \ll 1$ or $1 \ll \sigma S$.

If $E^{\frac{1}{2}} \ll \sigma S \ll 1$ and $\sigma S\alpha^2 \ll E$ $(-\frac{1}{2} < y < 0,\ y < -1-x)$ we may use (3.5a) and assume

$$v = v^{i0}(r,z) + \sigma S v^{i1}(r,z) + \sigma S\alpha^2 E^{-1} v^{i2}(r,z) + E^{\frac{1}{2}}(\sigma S)^{-1} v^{i3}(r,z), \tag{4.5a}$$

$$\psi = E(\sigma S)^{-1}\psi^{i0}(r) + E\psi^{i1}(r,z) + \alpha^2\psi^{i2}(r,z) + E^{\frac{3}{2}}(\sigma S)^{-2}\psi^{i3}(r,z), \tag{4.5b}$$

$$T = T^{i0}(r) + \sigma S T^{i1}(r,z) + \sigma S\alpha^2 E^{-1} T^{i2}(r,z) + E^{\frac{1}{2}}(\sigma S)^{-1} T^{i3}(r,z), \tag{4.5c}$$

$$b = 8b^{i0}(r,z) + \sigma S b^{i1}(r,z) + \sigma S\alpha^2 E^{-1} b^{i2}(r,z) + E^{\frac{1}{2}}(\sigma S)^{-1} b^{i3}(r,z), \tag{4.5d}$$

where v^{i0} is at most a linear function of z. In expansion (4.5) variables with superscript $i1$ become important as $\sigma S \to 1$, leading to the solutions presented in the second part of this subsection. Variables with superscript $i2$ become important as $\sigma S\alpha^2 \to E$, leading to the solutions presented in §4.4. Variables with superscript $i3$ become important as $\sigma S \to E^{\frac{1}{2}}$, leading to the solutions presented in §4.1.

Substitution of (4.5) into (2.2)–(2.5) yields

$$2v_z^{i0} = T_r^{i0}, \quad \psi^{i0} + T_r^{i0} = 0, \quad v_z^{i0} + 8(\nabla^2 - r^{-2})\, b^{i0} = 0. \tag{4.6}$$

In the interest of brevity the equations governing the variables with superscripts $i1$, $i2$ and $i3$ will not be presented. Their solutions are straightforward but complicated. The Ekman compatibility conditions for the dominant variables are

$$v^{i0} = \begin{pmatrix} v_T \\ v_B \end{pmatrix}, \quad b^{i0} = 0 \quad \text{at} \quad z = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$b^{i0} = 0 \quad \text{at} \quad r = r_0.$$

The solutions of (4.6) are

$$v^{i0} = \tfrac{1}{2}[v_B(r) + v_T(r)] + [v_T(r) - v_B(r)]\,(z - \tfrac{1}{2}), \tag{4.7a}$$

$$\psi^{i0} = -T_r^{i0} = 2[v_B(r) - v_T(r)]. \tag{4.7b}$$

With the insertion of the factor 8 in (4.5d), the solution for b^{i0} becomes (4.4).

As σS increases to unit order, the correction terms labelled $i1$ in (4.5) become important. Also, term [7] of (2.3) becomes important and (3.5a) is no longer valid. This is the case studied by Barcilon & Pedlosky (1967a). We shall not study the case $\sigma S = O(1)$ but proceed to the case $1 \ll \sigma S \ll E^{-1}$, $\alpha^2 \ll E$ $(0 < y < 1$, $x < -1)$. In this parameter region layers with scale $A5$ occur on $z = 0, 1$. Within these layers, $\partial/\partial z = O((\sigma S)^{\frac{1}{2}})$ and the variables have relative sizes

$$(v, \psi, T, b) = O((\sigma S)^{-1},\ E(\sigma S)^{-\frac{1}{2}},\ (\sigma S)^{-\frac{1}{2}},\ (\sigma S)^{-\frac{3}{2}}).$$

In order to satisfy all the boundary conditions on the end walls, layer $A5$ must have two separate scalings as follows:

$$v = v^{i0}(r,z) + (\sigma S)^{-1}[v^{i1}(r,z) + \hat{v}^1(r,\hat{z}) + \hat{\hat{v}}^1(r,\hat{\hat{z}})] + \alpha^2 E^{-1}[v^{i2}(r,z) + \hat{v}^2(r,\hat{z}) + \hat{\hat{v}}^2(r,\hat{\hat{z}})], \tag{4.8a}$$

$$\psi = E(\sigma S)^{-1}[\psi^{i1} + \hat{\psi}^1 + \hat{\hat{\psi}}^1] + \alpha^2(\sigma S)^{\frac{1}{2}}[\hat{\psi}^2 + \hat{\hat{\psi}}^2], \tag{4.8b}$$

$$T = T^i + (\sigma S)^{-\frac{1}{2}}[\hat{T}^1 + \hat{\hat{T}}^1] + (\sigma S)^{\frac{1}{2}}\alpha^2 E^{-1}[\hat{T}^2 + \hat{\hat{T}}^2], \tag{4.8c}$$

$$b = b^i + (\sigma S)^{-\frac{3}{2}}[\hat{b}^1 + \hat{\hat{b}}^1] + (\sigma S)^{-\frac{1}{2}}\alpha^2 E^{-1}[\hat{b}^2 + \hat{\hat{b}}^2], \tag{4.8d}$$

where

$$\hat{z} = z(\sigma S)^{\frac{1}{2}}, \quad \hat{\hat{z}} = (1 - z)(\sigma S)^{\frac{1}{2}}.$$

Variables in layer $A5$ near $z = 0$ $(z = 1)$ are denoted by a single (double) caret and superscripts 1 and 2 distinguish the two separate scalings. The interior variable v^{i1} becomes important as $\sigma S \to 1$, leading to the solutions presented in the first part of this subsection. The interior variable v^{i2} becomes important as $\alpha^2 \to E$, leading to the solutions presented in §4.3.

The equations governing the interior variables are, from (2.2)–(2.5),

$$(\nabla^2 - r^{-2})\, v^{i0} = 0, \quad T_r^i = 2v_z^{i0}, \tag{4.9a, b}$$

$$-(r\psi^i)_r/r = \nabla^2 T^i, \quad v_z^{i0} + (\nabla^2 - r^{-2})\, b^i = 0, \tag{4.9c, d}$$

$$(\nabla^2 - r^{-2})\, v^{i1} = 2\psi_z^i, \quad (\nabla^2 - r^{-2})\, v^{i2} = -2b_z^i. \tag{4.9e, f}$$

The equations for the lower $A5$ boundary layer on $z = 0$ are, suppressing the superscripts 1 and 2,

$$2\hat{v}_{\hat{z}} = \hat{T}_r, \quad 2\hat{\psi}_{\hat{z}} = \hat{v}_{\hat{z}\hat{z}}, \tag{4.10a, b}$$

$$-(r\hat{\psi})_r/r = \hat{T}_{\hat{z}\hat{z}}, \quad \hat{v}_{\hat{z}} + \hat{b}_{\hat{z}\hat{z}} = 0, \tag{4.10c, d}$$

while those for the layer on $z = 1$ are obtained from (4.10) by changing the sign of each term containing $\hat{v}$.

Since (4.9) and (4.10) are partial differential equations, we must specify conditions at $r = r_0$ as well as at $z = 0, 1$:

$$\left.\begin{aligned} v^{i0} = \begin{pmatrix} v_T \\ v_B \end{pmatrix}, \quad b^i = 0, \quad v^{i1} + \begin{pmatrix} \hat{v}^1 \\ \hat{\hat{v}}^1 \end{pmatrix} = 0 \\ T_z^i = -(rv^{i1})_r/2r, \quad v^{i2} + \begin{pmatrix} \hat{v}^2 \\ \hat{\hat{v}}^2 \end{pmatrix} = 0, \quad (rv^{i2})_r = 0 \end{aligned}\right\} \text{ at } z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{aligned} &(4.11\,a\text{–}c) \\ &(4.11\,d\text{–}f) \end{aligned}$$

$$\left.\begin{aligned} v^{i0} = v^{i1} = v^{i2} = 0, \quad b^i = 0 \\ \hat{T}_r^1 = \hat{\hat{T}}_r^1 = \hat{T}_r^2 = \hat{\hat{T}}_r^2 = 0 \end{aligned}\right\} \text{ at } r = r_0. \quad \begin{aligned} &(4.11\,g\text{–}h) \\ &(4.11\,i) \end{aligned}$$

In writing these conditions we have used the facts that $\hat{T}_{\hat{z}} = -(r\hat{v})_r/2r$ and $\hat{\hat{T}}_{\hat{\hat{z}}} = (r\hat{\hat{v}})_r/2r$.

The solution for v^{i0} satisfying (4.9 a) and conditions (4.11 a, g) is

$$v^{i0} = \sum_{n=1}^{\infty} \left\{ B_n^- \frac{\sinh[k_n(2z-1)/2r_0]}{\sinh[k_n/2r_0]} + B_n^+ \frac{\cosh[k_n(2z-1)/2r_0]}{\cosh[k_n/2r_0]} \right\} J_1\left(k_n \frac{r}{r_0}\right), \quad (4.12)$$

where $J_1(k_n) = 0$ and

$$B_n^{\pm} = \frac{1}{J_2^2(k_n)} \int_0^{r_0} \frac{\eta}{r_0^2} [v_T(\eta) \pm v_B(\eta)] J_1\left(k_n \frac{\eta}{r_0}\right) d\eta.$$

From (4.9 b) it is easily found that

$$T^i = -2 \sum_{n=1}^{\infty} \left\{ B_n^- \frac{\cosh[k_n(2z-1)/2r_0]}{\sinh[k_n/2r_0]} + B_n^+ \frac{\sinh[k_n(2z-1)/2r_0]}{\cosh[k_n/2r_0]} \right\} J_0\left(k_n \frac{r}{r_0}\right).$$

Since $\nabla^2 T^i = 0$ it follows from (4.9 c) that $\psi^i = 0$. Using (4.9 d) and (4.11 b, h) we have

$$b^i = \frac{1}{4} \sum_{n=1}^{\infty} \left\{ (B_n^- + B_n^+ - 2B_n^- z) \frac{\sinh[k_n(2z-1)/2r_0]}{\sinh[k_n/2r_0]} + (B_n^- + B_n^+ - 2B_n^+ z) \frac{\cosh[k_n(2z-1)/2r_0]}{\cosh[k_n/2r_0]} \right\} J_1\left(k_n \frac{r}{r_0}\right). \quad (4.13)$$

The solutions of (4.10) which satisfy the boundary conditions at $r = r_0$ are

$$\hat{v} = \sum_{n=1}^{\infty} \hat{v}_n \exp(-k_n \hat{z}/2r_0) J_1(k_n r/r_0), \quad (4.14\,a)$$

$$\hat{\psi} = \sum_{n=1}^{\infty} (-k_n \hat{v}_n/4r_0) \exp(-k_n \hat{z}/2r_0) J_1(k_n r/r_0), \quad (4.14\,b)$$

$$\hat{T} = \sum_{n=1}^{\infty} \hat{v}_n \exp(-k_n \hat{z}/2r_0) J_0(k_n r/r_0), \quad (4.14\,c)$$

$$\hat{b} = \sum_{n=1}^{\infty} (2r_0 \hat{v}_n/k_n) \exp(-k_n \hat{z}/2r_0) J_1(k_n r/r_0). \quad (4.14\,d)$$

With T^i and ψ^i known (4.9 e) may be solved for v^{i1} subject to conditions (4.11 d, g). The solution may be simply expressed as $v^{i1} = 4v^{i0}$. Next we may use condition (4.11 c) to determine that $\hat{v}_n^1 = 4(B_n^- - B_n^+)$ and $\hat{\hat{v}}_n^1 = -4(B_n^- + B_n^+)$. With b^i known (4.9 f) may be solved for v^{i2} subject to conditions (4.11 f, g). Finally, (4.11 e) may be used to determine $\hat{v}_n^2$ and $\hat{\hat{v}}_n^2$. These solutions will not be presented here.

4.3. *Solutions for parameter region* $\tilde{v}_{A2b}$

The dominant-order portions of the solutions presented in §§4.1 and 4.2 were known previously. Now we begin to present entirely new results. If $\alpha^2E^{-1} \ll \sigma S \ll E^{-1}$ and $E \ll \alpha^2$ $(x+1 < y < 1,\ -1 < x)$ layers with scale $A2b$ now occur on $z = 0, 1$ in addition to those with scale $A5$. The strength of the laminated flow is limited by electric currents to be of smaller than unit order. The size of the variables in layers $A2b$ is determined by $v = O(1)$ while the size in $A5$ is determined by the thermal boundary condition. Relying on (3.5*b*) we may assume

$$v = v^{i0}(r)+E^{\frac{1}{2}}\alpha^{-1}(z-\tfrac{1}{2})\,v^{i1}(r)+\bar{v}(r,\bar{z})+\bar{\bar{v}}(r,\bar{\bar{z}})+\alpha^2(\sigma SE)^{-1}[\hat{v}(r,\hat{z})+\hat{\hat{v}}(r,\hat{\hat{z}})], \quad (4.15a)$$

$$\psi = E^{\frac{3}{2}}(\sigma S\alpha)^{-1}\psi^i(r)+\alpha^3(\sigma S)^{-1}E^{-\frac{1}{2}}[\bar{\psi}+\bar{\bar{\psi}}]+\alpha^2(\sigma S)^{-\frac{1}{2}}[\hat{\psi}+\hat{\hat{\psi}}], \quad (4.15b)$$

$$T = E^{\frac{1}{2}}\alpha^{-1}T^i(r)+\alpha E^{-\frac{1}{2}}[\bar{T}+\bar{\bar{T}}]+\alpha^2(\sigma S)^{-\frac{1}{2}}E^{-1}[\hat{T}+\hat{\hat{T}}], \quad (4.15c)$$

$$b = E^{\frac{1}{2}}\alpha^{-1}[b^i(r)+\bar{b}+\bar{\bar{b}}]+\alpha^2(\sigma S)^{-\frac{3}{2}}E^{-1}[\hat{b}+\hat{\hat{b}}], \quad (4.15d)$$

where
$$\bar{z} = z\alpha E^{-\frac{1}{2}}, \quad \bar{\bar{z}} = (1-z)\,\alpha E^{-\frac{1}{2}}.$$

Variables in layer $A2b$ near $z = 0$ $(z = 1)$ are denoted by a single (double) overbar. As $\alpha^2 \to E$ the laminated-flow term v^{i1} becomes of dominant order, leading to the solutions presented in the second part of §4.2. As $\alpha^2 \to \sigma SE$ the azimuthal velocity in layers $A5$ becomes of unit order and layers $A2b$ and $A5$ merge, leading to solutions presented in §4.4.

Substitution of (4.15) into (2.2)–(2.5) yields the interior equations

$$2v^{i1} = T^i_r, \quad \psi^i+T^i_r = 0, \quad v^{i1}+[(rb^i)_r/r]_r = 0. \quad (4.16)$$

The equations governing the variables in the lower $A2b$ layer are

$$2\bar{b}_{\bar{z}}+\bar{v}_{\bar{z}\bar{z}} = 0, \quad \bar{v}_{\bar{z}}+\bar{b}_{\bar{z}\bar{z}} = 0, \quad 2\bar{v}_{\bar{z}} = \bar{T}_r, \quad (r\bar{\psi})_r/r+\bar{T}_{\bar{z}\bar{z}} = 0. \quad (4.17)$$

The equations for the upper $A2b$ layer are obtained from (4.17) by changing the sign of each term containing $\bar{v}$. The equations governing the variables in layers $A5$ are (4.10). The boundary conditions are

$$v^{i0}+\begin{pmatrix}\bar{\bar{v}}\\ \bar{v}\end{pmatrix} = \begin{pmatrix}v_T\\ v_B\end{pmatrix}, \quad b^i+\begin{pmatrix}\bar{\bar{b}}\\ \bar{b}\end{pmatrix} = 0, \quad \begin{pmatrix}\bar{\bar{T}}_{\bar{\bar{z}}}\\ \bar{T}_{\bar{z}}\end{pmatrix}+\begin{pmatrix}\hat{\hat{T}}_{\hat{\hat{z}}}\\ \hat{T}_{\hat{z}}\end{pmatrix} = 0 \quad \text{at} \quad z = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad (4.18a\text{–}c)$$

$$\hat{T}_r = \hat{\hat{T}}_r = b^i = 0 \quad \text{at} \quad r = r_0. \quad (4.18d)$$

The solutions of (4.17) satisfying conditions (4.18*b*) are

$$\bar{b} = -b^i(r)\exp(-2^{-\frac{1}{2}}\bar{z}), \quad \bar{v} = 2^{-\frac{1}{2}}b^i(r)\exp(-2^{-\frac{1}{2}}\bar{z}), \quad (4.19a, b)$$

$$\bar{T} = \left[c-\int_0^r b^i(\eta)\,d\eta\right]\exp(-2^{\frac{1}{2}}\bar{z}), \quad (4.19c)$$

where $b^i(r)$ is as yet unknown and c is an arbitrary constant of integration. The solutions for the upper $A2b$ layer are identical to (4.19) except for a change of sign of v and T.

Using (4.19*b*) and its counterpart from the upper layer in condition (4.18*a*) leads directly to

$$v^{i0}(r) = \tfrac{1}{2}[v_T(r)+v_B(r)], \quad b^i(r) = 2^{-\frac{1}{2}}[v_B(r)-v_T(r)]. \quad (4.20)$$

The interior azimuthal velocity is once again the average of the speeds of the top and bottom boundaries as conjectured at the end of §3.

From (4.16) we easily find that

$$2v^{i1} = -\psi^i = 2^{\frac{1}{2}}[(rv_T - rv_B)_r/r]_r, \quad T^i = 2^{\frac{1}{2}}(rv_T - rv_B)_r/r. \tag{4.21}$$

The solution of (4.10) satisfying (4.18 *d*) is given by (4.14). Now condition (4.18 *c*) requires that

$$\sum_{n=1}^{\infty} \frac{k_n \hat{v}_n}{2r_0} J_0\left(k_n \frac{r}{r_0}\right) + 2^{-\frac{1}{2}}\left[c - \int_0^r b^i(\eta)\, d\eta\right] = 0. \tag{4.22}$$

The first term of (4.22) has the form of a Dini series of Bessel functions (see Watson 1958, p. 580). For the present case, $H+\nu = 0$ in Watson's notation and the series should contain an initial term ($n = 0$). In its absence the constant c provides the necessary generality. Multiplying (4.22) by r and integrating from $r = 0$ to $r = r_0$ yields

$$c = \int_0^{r_0} \left(1 - \frac{\eta^2}{r_0^2}\right) b^i(\eta)\, d\eta.$$

The coefficients $\hat{v}_n$ are determined using equation (6) on p. 577 of Watson (1958):

$$\hat{v}_n = \frac{2^{\frac{3}{2}}}{k_n^2 J_0^2(k_n)} \int_0^{r_0} \eta b^i(\eta)\, J_1\left(k_n \frac{\eta}{r_0}\right) d\eta,$$

where $b^i(r)$ is given by (4.20 *b*).

4.4. *Solutions for parameter region* $\tilde{v}_{A7}$

As $\alpha^2 \to \sigma SE$ layers $A2b$ and $A5$ merge into a single layer labelled $A7$. The size of the variables in this layer is determined by the condition $v = O(1)$ provided that $\alpha^6 \ll \sigma SE \ll \alpha^2$ and $E \ll \sigma S\alpha^2$ $(3x+1 < y < x+1, \; -x-1 < y)$. Again relying on (3.5 *b*) we assume

$$v = v^{i0}(r) + (E/\sigma S\alpha^2)^{\frac{1}{2}}(z-\tfrac{1}{2})v^{i1}(r) + \tilde{v}(r,\tilde{z}) + \tilde{\tilde{v}}(r,\tilde{\tilde{z}}) + O((\sigma SE/\alpha^2)^{\frac{1}{2}}), \tag{4.23a}$$

$$\psi = (E/\sigma S)^{\frac{1}{2}}\alpha^{-\frac{1}{2}}\psi^i(r) + (E\alpha^6/\sigma S)^{\frac{1}{4}}[\tilde{\psi} + \tilde{\tilde{\psi}}], \tag{4.23b}$$

$$T = (E/\sigma S\alpha^2)^{\frac{1}{2}}T^i(r) + (\alpha^2\sigma S/E)^{\frac{1}{4}}[\tilde{T} + \tilde{\tilde{T}}], \tag{4.23c}$$

$$b = (E/\sigma S\alpha^2)^{\frac{1}{2}}[b^i(r) + \tilde{b} + \tilde{\tilde{b}}], \tag{4.23d}$$

where $$\tilde{z} = z(\sigma S\alpha^2/E)^{\frac{1}{4}}, \quad \tilde{\tilde{z}} = (1-z)(\sigma S\alpha^2/E)^{\frac{1}{4}}.$$

Variables in layer $A7$ near $z = 0$ $(z = 1)$ are denoted by a single (double) tilde. As $\sigma S\alpha^2 \to E$ the variable v^{i1} becomes of dominant order and layer $A7$ merges with the interior, leading to the solutions presented in the first part of §4.2. As $\alpha^6 \to \sigma SE$ the boundary-layer term in ψ becomes of order $E^{\frac{1}{2}}$, leading to solutions presented in §4.5. As $\sigma SE \to \alpha^2$ boundary-layer terms in v (not given) become of unit order, leading to the solutions presented in §4.3.

The interior equations are given by (4.16). The equations governing the variables in the lower $A7$ layer are

$$2\tilde{v}_{\tilde{z}} = \tilde{T}_r, \quad \tilde{\psi}_{\tilde{z}} = \tilde{b}_{\tilde{z}}, \quad -(r\tilde{\psi})_r/r = \tilde{T}_{\tilde{z}\tilde{z}}, \quad \tilde{v}_{\tilde{z}} + \tilde{b}_{\tilde{z}\tilde{z}} = 0, \tag{4.24 a–d}$$

while those for the upper $A7$ layer are obtained from (4.24) by changing the sign of each term containing $\tilde{v}$. The boundary conditions are

$$v^{i0}+\begin{pmatrix}\tilde{\tilde{v}}\\ \tilde{v}\end{pmatrix}=\begin{pmatrix}v_T\\ v_B\end{pmatrix},\quad b^i+\begin{pmatrix}\tilde{\tilde{b}}\\ \tilde{b}\end{pmatrix}=0,\quad \begin{pmatrix}\tilde{\tilde{T}}_{\tilde{\tilde{z}}}\\ \tilde{T}_{\tilde{z}}\end{pmatrix}=0 \quad \text{at} \quad z=\begin{pmatrix}1\\ 0\end{pmatrix}, \tag{4.25a–c}$$

$$\tilde{b}=\tilde{\tilde{b}}=0 \quad \text{at} \quad r=r_0. \tag{4.25d}$$

The boundary-layer problems for $\tilde{b}$ and $\tilde{\tilde{b}}$ may both be expressed as

$$[(rb)_r/r]_r = 2b_{zzzz}, \tag{4.26a}$$

$$b=-b^i(r),\quad b_{zzz}=0 \quad \text{at} \quad z=0, \tag{4.26b}$$

$$b\to 0 \quad \text{as} \quad z\to\infty, \tag{4.26c}$$

$$b=0 \quad \text{at} \quad r=r_0, \tag{4.26d}$$

where the tildes have been omitted. This problem has solution

$$b=\sum_{n=1}^{\infty} A_n \exp(-\lambda_n z)\,[\cos\lambda_n z-\sin\lambda_n z]\,J_1(k_n r/r_0), \tag{4.27}$$

where $J_1(k_n)=0$, $\lambda_n=(k_n)^{\frac{1}{2}}\,2^{-\frac{3}{4}}\,r_0^{-\frac{1}{2}}$ and

$$A_n=\frac{-2}{J_2^2(k_n)}\int_0^{r_0}\frac{\eta}{r_0^2}\,b^i(\eta)\,J_1\left(k_n\frac{\eta}{r_0}\right)d\eta.$$

Note that the conditions $\tilde{b}=\tilde{\tilde{b}}=0$ at $r=r_0$ together with (4.24) ensure that $\tilde{\psi}=\tilde{\tilde{\psi}}=\tilde{T}_r=\tilde{\tilde{T}}_r=0$ at $r=r_0$. This allows the side-wall and end-wall boundary layers to be decoupled.

Since the problems for $\tilde{b}$ and $\tilde{\tilde{b}}$ are identical it follows that $\tilde{b}=\tilde{\tilde{b}}$. Noting that $\partial/\partial\tilde{z}=-\partial/\partial\tilde{\tilde{z}}$ we see from (4.24) that $\tilde{v}=-\tilde{\tilde{v}}$. This fact allows us to add the upper and lower conditions (4.25*a*) and arrive at the important result

$$v^{i0}(r)=\tfrac{1}{2}[v_T(r)+v_B(r)] \tag{4.28}$$

once more. Now from either of conditions (4.25*a*)

$$\tilde{v}(r,0)=-\tilde{\tilde{v}}(r,0)=\tfrac{1}{2}[v_B(r)-v_T(r)]$$

and $\tilde{b}$ is given by (4.27), where

$$A_n=\frac{1}{2J_2^2(k_n)}\int_0^{r_0}\frac{\eta}{r_0^2}[v_T(\eta)-v_B(\eta)]\,J_1\left(k_n\frac{\eta}{r_0}\right)d\eta.$$

From (4.25*b*) we have

$$b^i=-\sum_{n=1}^{\infty}A_n J_1(k_n r/r_0),$$

which can be summed using equations (3) and (4) on p. 576 of Watson (1958) to read

$$b^i=\tfrac{1}{4}[v_B(r)-v_T(r)]. \tag{4.29}$$

Finally, from (4.16) we have

$$2v^{i1}=-\psi^i=\tfrac{1}{2}[(rv_T-rv_B)_r/r]_r,\quad T^i=\tfrac{1}{2}(rv_T-rv_B)_r/r. \tag{4.30}$$

4.5. *Solutions for parameter region* $\tilde{\psi}_{A7}$

If $E\alpha^{-2} \ll \sigma S \ll \alpha^6 E^{-1}$ and $\alpha^2 \ll 1$ $(-1-x < y < 3x+1,\ x < 0)$ the size o the variables in layer $A7$ is determined by the condition $\psi = O(E^{\frac{1}{2}})$. Relying on (3.5*b*) once more we assume

$$v = v^{i0}(r) + E^{\frac{1}{2}}\alpha^{-2}v^{i1}(r)\,(z-\tfrac{1}{2}) + (\sigma SE/\alpha^6)^{\frac{1}{4}}[\tilde{v}(r,\tilde{z}) + \tilde{\tilde{v}}(r,\tilde{\tilde{z}})], \quad (4.31\,a)$$

$$\psi = E^{\frac{3}{2}}(\sigma S)^{-1}\alpha^{-2}\psi^i(r) + E^{\frac{1}{2}}[\tilde{\psi} + \tilde{\tilde{\psi}}], \quad (4.31\,b)$$

$$T = E^{\frac{1}{2}}\alpha^{-2}T^i(r) + (\sigma S)^{\frac{1}{2}}\alpha^{-1}[\tilde{T} + \tilde{\tilde{T}}], \quad (4.31\,c)$$

$$b = E^{\frac{1}{2}}\alpha^{-2}[b^i(r) + \tilde{b} + \tilde{\tilde{b}}], \quad (4.31\,d)$$

where $\tilde{z}$ and $\tilde{\tilde{z}}$ are given following (4.23). As $\sigma SE \to \alpha^6$ the variables $\tilde{v}$ and $\tilde{\tilde{v}}$ become of unit order, leading to solutions presented in §4.4. As $\sigma S\alpha^2 \to E$ layer $A7$ thickens to merge with the interior and we have the case considered in §4.1. As $\alpha^2 \to 1$, $v_z = O(E^{\frac{1}{2}})$ and assumption (3.7) is no longer valid.

The interior equations are given by (4.16) and the $A7$-layer equations by (4.24). The boundary conditions are

$$\begin{pmatrix}\tilde{\tilde{\psi}}\\ \tilde{\psi}\end{pmatrix} = \begin{pmatrix}+\\ -\end{pmatrix}\frac{1}{2}\left[v^{i0} - \begin{pmatrix}v^T\\ v^B\end{pmatrix}\right], \quad b^i + \begin{pmatrix}\tilde{\tilde{b}}\\ \tilde{b}\end{pmatrix} = 0, \quad \begin{pmatrix}\tilde{\tilde{T}}_{\tilde{\tilde{z}}}\\ \tilde{T}_{\tilde{z}}\end{pmatrix} = 0 \quad \text{at} \quad z = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad (4.32\,a\text{–}c)$$

$$\tilde{b} = \tilde{\tilde{b}} = 0 \quad \text{at} \quad r = r_0. \quad (4.32d)$$

The problems for $\tilde{b}$ and $\tilde{\tilde{b}}$ are again identical, given by (4.26). From (4.24*b*) we obtain $\tilde{\psi}(r,0) = \tilde{\tilde{\psi}}(r,0)$. Using (4.32*a*) we again have for the interior flow

$$v^{i0}(r) = \tfrac{1}{2}[v_B(r) + v_T(r)]. \quad (4.33)$$

Also from the boundary conditions we have (4.29) for b_i. The solutions for v^{i1}, T^i and ψ^i are given by (4.30).

5. Solutions for side-wall boundary layers

The purpose of this and the following section is to demonstrate that we have scaled the problem correctly and that we can find side-wall boundary-layer solutions to close the problem for each parameter region. In this section we shall present the scaling, equations and general solutions for each of the side-wall layers which may occur in the parameter range $E \ll \sigma S \ll E^{-1}$, $\alpha^2 \ll 1$: $B1$, $B2b$, $B4$, $B5$, $B7$ and $B8$, where we have included the $E^{\frac{1}{4}}$ side-wall layer as layer $B8$. Of these layers only $B7$ is new and, as we shall see, its solution is surprisingly difficult and complicated. Following this catalogue of layer solutions we shall study the closure problem in §6. This consists of forming linear combinations of the solutions for the side-wall layers, which, when added to the interior solutions found in §4, satisfy the side-wall boundary conditions. These sections are primarily of technical interest and could be skipped without loss of continuity.

Before proceeding with the catalogue of side-wall boundary-layer solutions let us simplify the problem. It was argued in §3 that (2.4) may be replaced by (3.6) in the interior and in all side-wall boundary layers. Also it was seen that term [3] of (2.2) is negligibly small in the interior. From figure 2 and table 1 it may be seen that term [3] of (2.2) is also small in any layer which exists for $\alpha^2 \ll 1$. Consequently we may eliminate the thermal perturbation T from direct consideration and reduce the interior and side-wall problem to

$$2v_z = -\sigma S E^{-1}\psi - E\psi_{rrrr}, \quad 2\psi_z = 2\alpha^2 b_z + E v_{rr}, \tag{5.1), (5.2}$$

$$v_z + (\nabla^2 - r^{-2})b = 0, \tag{5.3}$$

with conditions

$$v = \psi = \psi_r = b = 0 \quad \text{at} \quad r = r_0. \tag{5.4a–d}$$

5.1. *Layer B1: the Stewartson $E^{\frac{1}{3}}$ layer*

In layer $B1$ the variables have relative sizes

$$(v, \psi, b) = O(E^{\frac{1}{6}}, E^{\frac{1}{2}}, E^{\frac{5}{6}}). \tag{5.5}$$

The governing equations for the scaled variables are

$$2v_z = -\psi_{\tau\tau\tau\tau}, \quad 2\psi_z = v_{\tau\tau}, \quad v_z + b_{\tau\tau} = 0, \tag{5.6}$$

where $\tau = (r_0 - r)E^{-\frac{1}{3}}$.

Equations (5.6) are partial differential equations which allow satisfaction of one boundary condition at $z = 0$ and one at $z = 1$. From (5.5) we see that ψ is of the same order as in the interior while v and b in layer $B1$ are smaller than in the interior. Therefore we specify $\psi = 0$ at $z = 0, 1$ and obtain the general solution

$$\begin{aligned} v = \sum_{n=1}^{\infty} \tfrac{1}{2}(2n\pi)^{\frac{1}{3}} [2\psi_n e^{-2\tau_n} + (3^{\frac{1}{2}}\psi_n^s - \psi_n^c)\cos(3^{\frac{1}{2}}\tau_n)e^{-\tau_n} \\ + (-\psi_n^s - 3^{\frac{1}{2}}\psi_n^c)\sin(3^{\frac{1}{2}}\tau_n)e^{-\tau_n}]\cos(n\pi z), \end{aligned} \tag{5.7a}$$

$$\psi = \sum_{n=1}^{\infty} [\psi_n e^{-2\tau_n} + \psi_n^c \cos(3^{\frac{1}{2}}\tau_n)e^{-\tau_n} + \psi_n^s \sin(3^{\frac{1}{2}}\tau_n)e^{-\tau_n}]\sin(n\pi z), \tag{5.7b}$$

$$\begin{aligned} b = \sum_{n=1}^{\infty} (n\pi/4)^{\frac{2}{3}} [2\psi_n e^{-2\tau_n} - (3^{\frac{1}{2}}\psi_n^s + \psi_n^c)\cos(3^{\frac{1}{2}}\tau_n)e^{-\tau_n} \\ + (3^{\frac{1}{2}}\psi_n^c - \psi_n^s)\sin(3^{\frac{1}{2}}\tau_n)e^{-\tau_n}]\sin n\pi z, \end{aligned} \tag{5.7c}$$

where $\tau_n = \frac{1}{2}(2n\pi)^{\frac{1}{3}}\tau$.

5.2. *Layer B2b: the hydromagnetic side-wall layer*

In layer $B2b$ the variables have relative sizes

$$(v, \psi, b) = O(E^0, E(\sigma S)^{-1}, E^{\frac{1}{2}}\alpha^{-1}). \tag{5.8}$$

The governing equations are

$$2^{\frac{1}{2}}b_z + v_{\omega\omega} = 0, \quad v_z + 2^{\frac{1}{2}}b_{\omega\omega} = 0, \quad \psi = -2v_z, \tag{5.9}$$

where $\omega = (r_0 - r)(2\alpha^2/E)^{\frac{1}{2}}$.

This problem is similar to that presented in Roberts (1967, pp. 186–189). If we define $\Gamma = v + 2^{\frac{1}{2}}b$ and $\Phi = v - 2^{\frac{1}{2}}b$, we obtain uncoupled equations

$$\Gamma_z + \Gamma_{\omega\omega} = 0, \quad \Phi_z - \Phi_{\omega\omega} = 0, \tag{5.10}$$

which are parabolic.

The solutions of (5.10) may be expressed as

$$\Gamma = \frac{2}{\sqrt{\pi}}\int_{\frac{1}{2}\omega(1-z)^{-\frac{1}{2}}}^{\infty} \Gamma_0(z-\omega^2/4\eta^2)\, e^{-\eta^2}\, d\eta, \tag{5.11 a}$$

$$\Phi = \frac{2}{\sqrt{\pi}}\int_{\frac{1}{2}\omega z^{-\frac{1}{2}}}^{\infty} \Phi_0(z-\omega^2/4\eta^2)\, e^{-\eta^2}\, d\eta \tag{5.11 b}$$

(see Carslaw & Jaeger 1959, pp. 62–63), where $\Gamma_0(z) = \Gamma(0,z)$ and $\Phi_0(z) = \Phi(0,z)$.

If Γ_0 and Φ_0 are constant then the solutions reduce to

$$\Gamma = \Gamma_0 \operatorname{erfc}[\tfrac{1}{2}\omega(1-z)^{-\frac{1}{2}}], \quad \Phi = \Phi_0 \operatorname{erfc}[\tfrac{1}{2}\omega z^{-\frac{1}{2}}]. \tag{5.12}$$

5.3. *Layer B4: the buoyancy layer*

Layer $B4$ is the buoyancy layer, with scaling

$$(v, \psi, b) = O(E^{\frac{1}{2}}(\sigma S)^{-\frac{1}{2}}, E^{\frac{1}{2}}, E^{\frac{3}{4}}(\sigma S)^{-1}). \tag{5.13}$$

The governing equations are

$$\psi_{\gamma\gamma\gamma\gamma} + 4\psi = 0, \quad v_{\gamma\gamma} = 4\psi_z, \quad b_{\gamma\gamma} = -2v_z, \tag{5.14}$$

where $\gamma = (r_0 - r)(\sigma S/4E^{\frac{1}{2}})^{\frac{1}{4}}$. Note that if $1 \ll \sigma S$ this layer is thinner than the Ekman layer. These are ordinary differential equations with simple negative exponential solutions

$$v = [2\psi_z^s \cos\gamma - 2\psi_z^c \sin\gamma]\, e^{-\gamma}, \tag{5.15 a}$$

$$\psi = [\psi^c \cos\gamma + \psi^s \sin\gamma]\, e^{-\gamma}, \tag{5.15 b}$$

$$b = [2\psi_{zz}^c \cos\gamma + 2\psi_{zz}^s \sin\gamma]\, e^{-\gamma}. \tag{5.15 c}$$

Since layer $B4$ is always the thinnest side-wall layer when it occurs, it usually must satisfy the condition $\psi_\gamma = 0$ at $\gamma = 0$. This requires

$$\psi^c = \psi^s. \tag{5.16}$$

Solution (5.15) with (5.16) is equivalent to that for the buoyancy layer presented in Barcilon & Pedlosky (1967*b*).

5.4. *Layer B5: the hydrostatic layer*

For the variables in the hydrostatic layer $B5$ we present two sets of scaling:

$$(v, \psi, b) = O(\sigma S E^{-\frac{1}{2}}, E^{\frac{1}{2}}, (\sigma S)^2 E^{-\frac{1}{2}}) \tag{5.17 a}$$

or

$$(v, \psi, b) = O(E^0, E(\sigma S)^{-1}, \sigma S). \tag{5.17 b}$$

In either case the equations are

$$-2v_z = \psi, \quad 2\psi_z = v_{\theta\theta}, \quad b_{\theta\theta} = -v_z, \tag{5.18}$$

where $\theta = (r_0 - r)(\sigma S)^{-\frac{1}{2}}$.

If $\sigma S \ll E^{\frac{1}{2}}$ we use scaling (5.17 *a*) and, following the arguments above for layer $B1$, specify $\psi = 0$ at $z = 0, 1$. In this case the general solution is

$$v = \sum_{n=1}^{\infty} (2n\pi)^{-1} \psi_n \exp(-2n\pi\theta) \cos(n\pi z), \tag{5.19 a}$$

$$\psi = \sum_{n=1}^{\infty} \psi_n \exp(-2n\pi\theta) \sin(n\pi z), \tag{5.19 b}$$

$$b = \sum_{n=1}^{\infty} (8n^2\pi^2)^{-1} \psi_n \exp(-2n\pi\theta) \sin(n\pi z). \tag{5.19 c}$$

On the other hand if $E^{\frac{1}{2}} \ll \sigma S$ scaling (5.17 *b*) is applicable and we must specify $v = 0$ at $z = 0, 1$. Now the general solution is

$$v = \sum_{n=1}^{\infty} v_n \exp(-2n\pi\theta) \sin(n\pi z), \tag{5.20 a}$$

$$\psi = \sum_{n=1}^{\infty} -2n\pi v_n \exp(-2n\pi\theta) \cos(n\pi z), \tag{5.20 b}$$

$$b = \sum_{n=1}^{\infty} -(4n\pi)^{-1} v_n \exp(-2n\pi\theta) \cos(n\pi z). \tag{5.20 c}$$

5.5. *Layer B7: a new side-wall boundary layer*

Layer $B7$ is the only new side-wall layer to come out of the scale analysis in appendix A. The equations for this layer may be written as

$$2v_z = -\sigma S E^{-1}\psi, \quad \psi_z = \alpha^2 b_z, \quad v_z + (\sigma S \alpha^2 E^{-1}/8)\, b_{\delta\delta} = 0, \tag{5.21 a–c}$$

where $\delta = (\sigma S\alpha^2/8E)^{\frac{1}{2}}(r_0 - r)$ represents a stretched radial co-ordinate but the variables v, ψ and b are *unscaled*. In terms of a single variable (5.21) reduces to

$$b_{\delta\delta z} = 4b_z. \tag{5.22}$$

Layer $B7$ exists to satisfy the condition $b + b^i = 0$ at the side wall. Whenever layer $B7$ occurs, b^i is independent of z [see (4.23) and (4.31)]. If we assume that b is independent of z as well, the equations governing layer $B7$ [either (5.21) or (5.22)] become degenerate owing to the presence of the z derivatives in (5.21 *b*). In this case the thickness and structure of the layer are governed by the boundary conditions at $z = 0$ and $z = 1$ in a manner similar to that for the $E^{\frac{1}{4}}$ layer (see §5.6). Whereas the $E^{\frac{1}{4}}$ layer is determined solely by consideration of the Ekman compatibility conditions at the end walls, in the present case it is necessary to consider the corner regions which exist above and below layer $B7$ as well, resulting in a complicated analysis which has been relegated to appendix B. The result of that analysis is the following differential equation governing the variation of b in layer $B7$:

$$(1 + \sigma S E^{-\frac{1}{2}}/8)\, b_{\delta\delta} = [4 + 2^{\frac{3}{2}}(\sigma S)^{\frac{1}{2}}\alpha^{-1}]\, b. \tag{5.23}$$

The solution of (5.23) divides into three separate cases.

If $\sigma S \ll E^{\frac{1}{2}}$ and $E \ll \sigma S\alpha^2$ it follows that $\sigma S \ll \alpha^2$ and (5.23) is simply

$$b_{\delta\delta} = 4b. \tag{5.24}$$

In this case layer $B7$ exists with the predicted scale as layer $B7a$ and

$$b = -b^i(r) \exp[-(\sigma S\alpha^2/2E)^{\frac{1}{2}}(r_0 - r)], \tag{5.25 a}$$

$$v = \tfrac{1}{4}\sigma S\alpha^2 E^{-1}(1-2z)\, b, \quad \psi = \alpha^2 b. \tag{5.25 b, c}$$

If $E^{\frac{1}{2}} \ll \sigma S \ll \alpha^2$, (5.23) becomes

$$(\sigma S E^{-\frac{1}{2}}/32)\, b_{\delta\delta} = b. \tag{5.26}$$

In this case layer $B7$, as layer $B7b$ has a greater horizontal scale than predicted and

$$b = -b^i(r_0) \exp[-(2\alpha/E^{\frac{1}{4}})(r_0 - r)], \tag{5.27 a}$$

$$v = 2\alpha^2 E^{-\frac{1}{2}}(1-2z)\, b, \quad \psi = 8E^{\frac{1}{2}}\alpha^2(\sigma S)^{-1} b. \tag{5.27 b, c}$$

If $\alpha^2 \ll \sigma S$ it follows that $E^{\frac{1}{2}} \ll \sigma S$ and (5.23) becomes

$$(\tfrac{1}{2}\sigma S\alpha^2 E^{-1})^{\frac{1}{2}} b_{\delta\delta} = 16b. \tag{5.28}$$

In this case layer $B7$, as layer $B7\,c$, has the same scale as $A7$ and

$$b = -b^i(r_0)\exp[-2(\sigma S\alpha^2/2E)^{\frac{1}{4}}(r_0-r)], \tag{5.29a}$$

$$v = (2\sigma S\alpha^2/E)^{\frac{1}{2}}(1-2z)b, \quad \psi = 4(2\alpha^2 E/\sigma S)^{\frac{1}{2}}b. \tag{5.29b,c}$$

5.6. *Layer B8: the Stewartson $E^{\frac{1}{4}}$ layer*

The $E^{\frac{1}{4}}$ layer must be handled with some care because the azimuthal velocity in this layer has a correction term which is as large as the laminated flow. Following Barcilon & Pedlosky (1967*b*), let

$$v(r,z) = v^{(0)}(\xi,z) + \sigma S E^{-\frac{1}{2}} v^{(1)}(\xi,z) + \ldots, \tag{5.30a}$$

$$\psi(r,z) = E^{\frac{1}{2}}\psi^{(0)}(\xi,z) + \sigma S\psi^{(1)}(\xi,z) + \ldots, \tag{5.30b}$$

$$b(r,z) = \sigma S b^{(0)}(\xi,z) + \ldots, \tag{5.30c}$$

where $\xi = (r_0 - r)E^{-\frac{1}{4}}[1 + \lambda\sigma S E^{-\frac{1}{2}} + \ldots]$.

Equations (5.1)–(5.3) become

$$v_z^{(0)} = 0, \quad 2v_z^{(1)} = -\psi^{(0)}, \quad 2\psi_z^{(0)} = v_{\xi\xi}^{(0)}, \tag{5.31a–c}$$

$$2\psi_z^{(1)} = v_{\xi\xi}^{(1)} + 2\lambda v_{\xi\xi}^{(0)}, \quad v_z^{(1)} = -b_{\xi\xi}^{(0)}. \tag{5.31d,e}$$

These equations are insufficient to determine the solutions, thus we must specify the Ekman compatibility conditions

$$\psi^{(0)} = \pm\tfrac{1}{2}v^{(0)}, \quad \psi^{(1)} = \pm\tfrac{1}{2}v^{(1)} \quad \text{at} \quad z = \tfrac{1}{2}\pm\tfrac{1}{2}. \tag{5.32}$$

This set of equations has solution

$$v^{(0)} = v_0\exp(-2^{\frac{1}{2}}\xi), \quad \psi^{(0)} = b^{(0)} = v_0(z-\tfrac{1}{2})\exp(-2^{\frac{1}{2}}\xi), \tag{5.33a,b}$$

$$\psi^{(1)} = [-\tfrac{1}{12}v_0(z-\tfrac{1}{2})^2 - \tfrac{1}{24}v_0 + v_1](z-\tfrac{1}{2})\exp(-2^{\frac{1}{2}}\xi), \tag{5.33c}$$

$$v^{(1)} = [-\tfrac{1}{4}v_0(z-\tfrac{1}{2})^2 + v_1]\exp(-2^{\frac{1}{2}}\xi), \quad \lambda = -\tfrac{1}{48}. \tag{5.33d,e}$$

This completes the catalogue of the side-wall boundary-layer solutions. We now turn to our final task of combining the solutions found above with the interior solutions found in §4 to satisfy conditions (5.4) to dominant order.

6. The closure problem

The analysis of this section may be simplified by noting that for the parameter region $E^{\frac{2}{3}} \ll \sigma S$ the thermal-wind balance is valid in the interior and in side-wall layers $B2b$, $B5$, $B7$ and $B8$. From (5.1) the thermal-wind balance may be expressed as

$$2v_z = -\sigma S E^{-1}\psi. \tag{6.1}$$

This equation is not valid within the boundary layer $B4$, which exists to satisfy the conditions on ψ and ψ_r at the side wall. However, whenever we satisfy the condition $v = 0$ at $r = r_0$ by some combination of solutions for the side-wall layers $B2b$, $B5$, $B7$ and $B8$ together with the interior, (6.1) shows that the condition

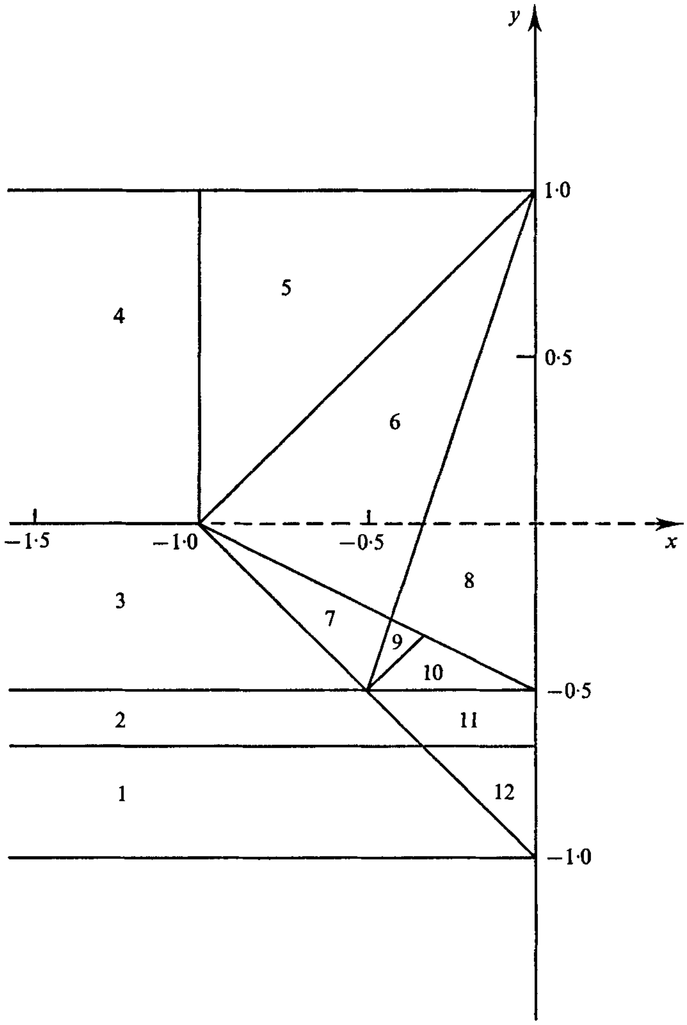

FIGURE 4. The x, y plane, showing the twelve regions in which different interior and side-wall-layer combinations occur. The layers occurring for each region are listed in table 4.

$\psi = 0$ at $r = r_0$ is automatically satisfied. It follows that *boundary layer B*4 *is absent to dominant order*. It must be present at higher order to ensure that $\psi_r = 0$ at $r = r_0$ but we shall ignore those solutions. At the same time we may discard the side-wall boundary conditions on ψ and ψ_r and seek to satisfy only

$$v = b = 0 \quad \text{at} \quad r = r_0 \tag{6.2}$$

by some combination of the solutions in §§5.2 and 5.4–5.6 together with the interior solutions from §§4.2–4.5. This simplification applies only if $E^{\frac{2}{3}} \ll \sigma S$; if $\sigma S \ll E^{\frac{2}{3}}$ we must find a linear combination of the solutions in §§5.1 and 5.5–5.6 together with interior solutions from §4.1 or §4.5 which satisfy conditions (5.4). The closure problem must be repeated for each region in the parameter plane for which a different set of side-wall layers occurs or a different interior solution is valid. Taking into account the three possible forms for layer $B7$, we

Region number	Side-wall layers	Interior solution
1	$B1$, $B8$	ψ^i
2	($B4$), $B5$, $B8$	
3	($B4$), $B5$	v^i
4	($B4$)	
5	$B2b$, ($B4$)	$\tilde{v}_{A2b}$
6	$B2b$, ($B4$)	$\tilde{v}_{A7}$
7	($B4$), $B5$, $B7c$	
8	$B2b$, ($B4$)	
9	($B4$), $B5$, $B7c$	
10	($B4$), $B5$, $B7b$	$\tilde{\psi}_{A7}$
11	($B4$), $B5$, $B7a$, $B8$	
12	$B1$, $B7a$, $B8$	

TABLE 4. A list of the side-wall-layer and interior solutions for each of the twelve parameter regions labelled in figure 4. Side-wall layer $B4$ is indicated in parentheses since it does not occur to dominant order for the particular side-wall boundary conditions specified in this problem.

have a total of twelve parameter regions to consider. These regions are drawn on figure 4 whilethe side-wall layers and solutions for each region are listed in table 4.

(1) If $E \ll \sigma S \ll E^{\frac{2}{3}}$ and $\sigma S\alpha^2 \ll E$ we have classic homogeneous non-magnetic rotating flow to dominant order. The interior variables are given by (4.1) and (4.3). In this parameter region only side-wall layers $B1$ and $B8$ can exist. They are capable of satisfying conditions (5.4*a–c*). These solutions are (5.7) and (5.33) with

$$\psi_n^c = \{v_0[1+(-1)^n]+[v_B(r_0)-v_T(r_0)]\,[1-(-1)^n]\}/n\pi, \qquad (6.3a)$$

$$\psi_n^s = 3^{-\frac{1}{2}}\psi_n^c, \quad \psi_n = 0, \qquad (6.3b,c)$$

$$v_0 = -\tfrac{1}{2}[v_B(r_0)+v_T(r_0)]. \qquad (6.3d)$$

The solution for b is given entirely by (4.4), which satisfies (5.4*d*) directly. Whenever the $E^{\frac{1}{4}}$ layer $B8$ occurs we have the freedom to satisfy the condition $v = 0$ at $r = r_0$ to dominant order and to the next smaller order. If $\sigma S \ll E^{\frac{3}{4}}$ the next smaller order is $v = O(E^{\frac{1}{6}})$, generated by the $B1$ side-wall layer; this condition is satisfied by (6.3). However, if $E^{\frac{3}{4}} \ll \sigma S$ and $\sigma S\alpha^2 \ll E$ the next smaller order is $v = O(\sigma SE^{-\frac{1}{2}})$, generated by the interior and the $B8$ layer. This causes a change in the boundary-layer structure and leads to case (2).

(2) If $E^{\frac{3}{4}} \ll \sigma S \ll E^{\frac{1}{2}}$ and $\sigma S\alpha^2 \ll E$ layer $B1$ splits into a thinner buoyancy layer $B4$, which we shall ignore, and a thicker hydrostatic layer $B5$. The interior variables are still given by (4.1) and (4.3) but now we may satisfy conditions (6.2) rather than (5.4). As before the function of layer $B8$ is to satisfy the dominant condition on v. This is given by (5.33), where

$$v_0 = -\tfrac{1}{2}[v_T(r_0)+v_B(r_0)]. \qquad (6.4)$$

To next order ($\sigma S E^{-\frac{1}{2}}$) the azimuthal velocity at $r = r_0$ has contributions from the interior and from layers $B5$ and $B8$:

$$[v_T(r_0) - v_B(r_0)]\,(z-\tfrac{1}{2}) + \sum_{n=1}^{\infty} 4(n\pi)^{-1}\psi_n \cos(n\pi z) + [v_T(r_0)+v_B(r_0)]\,(z-\tfrac{1}{2})^2 + 8v_1 = 0, \quad (6.5)$$

where we have used (5.19). This equation is equivalent to equation (5.11) of Barcilon & Pedlosky (1967*b*). From the components of (6.5) we obtain

$$v_1 = -\tfrac{1}{96}[v_T(r_0)+v_B(r_0)], \quad \psi_n = -[(-1)^n v_T(r_0) + v_B(r_0)]\,(n\pi)^{-1}. \quad (6.6)$$

The solution for b remains (4.4). It may easily be verified that these solutions satisfy $\psi = 0$ at $r = r_0$ to $O(E^{\frac{1}{2}})$.

(3) If $E^{\frac{1}{2}} \ll \sigma S \ll 1$ and $\sigma S \alpha^2 \ll E$ laminated flow dominates (i.e. $v_z^i = O(1)$) and the interior variables are given by (4.5), (4.7) and (4.4). In this parameter region the $E^{\frac{1}{4}}$ layer $B8$ has been absorbed into the hydrostatic layer $B5$, which now satisfies the dominant boundary condition on v. The solution for layer $B5$ is given by (5.20) with

$$v_n = [-v_B(r_0) + (-1)^n v_T(r_0)]\,(2n\pi)^{-1}. \quad (6.7)$$

Again b is given by (4.4), which satisfies (6.2*b*).

(4) If $1 \ll \sigma S \ll E^{-1}$ and $\alpha^2 \ll E$ the interior flow is controlled by viscous diffusion. The interior variables, which are given by (4.8), (4.12) and (4.13), directly satisfy conditions (6.2) to dominant order and no side-wall layers are necessary in this parameter region.

This completes the problem for all non-magnetic regions with $E \ll \sigma S \ll E^{-1}$. We have considered the four regions at the extreme left on figure 4. We now proceed to investigate the regions where magnetic effects play an important role.

(5) If $\alpha^2 \ll \sigma SE \ll 1$ and $E \ll \alpha^2$ the relevant interior variables are given by (4.15) and (4.20). In this parameter range layer $B2b$ exists to satisfy the side-wall conditions (6.2). With scaling (5.8) the solutions for layer $B2b$ are given by (5.12), where

$$\Gamma_0 = \tfrac{3}{2}v_B(r_0) - \tfrac{1}{2}v_T(r_0), \quad \Phi = \tfrac{3}{2}v_T(r_0) - \tfrac{1}{2}v_B(r_0). \quad (6.8)$$

(6) If $\alpha^6 \ll \sigma SE \ll \alpha^2$ and $E \ll (\sigma S)^2\alpha^2$ the relevant interior variables are given by (4.23), (4.28) and (4.29). In order to satisfy condition (6.2*b*) in this parameter region we must use the scaling

$$(v, b) = O(\alpha^{\frac{1}{2}}(\sigma SE)^{-\frac{1}{4}},\ E^{\frac{1}{4}}(\sigma S)^{-\frac{1}{4}}\alpha^{\frac{1}{2}}) \quad (6.9)$$

in place of (5.8) for layer $B2b$. Note that this generates a large azimuthal velocity within layer $B2b$. With the scaling (6.9) the solution for layer $B2b$ is given by (5.12), where

$$\Gamma_0 = -\Phi_0 = 2^{-\frac{3}{2}}[v_T(r_0) - v_B(r_0)]. \quad (6.10)$$

(7) If $\sigma SE \ll (\sigma S)^2\alpha^2 \ll E$ and $\alpha^6 \ll \sigma SE$ the interior variables are still given by (4.23), (4.28) and (4.29) but layer $B2b$ has bifurcated into layers $B5$ and $B7c$, which exist to satisfy the conditions on v and b respectively. In order to satisfy condition (6.2*b*) in this parameter region we must use the scaling

$$(v, b) = O((\sigma S)^{\frac{1}{4}}\alpha^{\frac{1}{2}}E^{-\frac{1}{4}},\ E^{\frac{1}{4}}(\sigma S)^{-\frac{1}{4}}\alpha^{-\frac{1}{2}}) \quad (6.11)$$

for layer $B7c$. Again this generates a large azimuthal velocity† within layer $B7c$, which requires layer $B5$ to exist with scaling

$$(v, \psi, b) = O((\sigma S)^{\frac{1}{4}} \alpha^{\frac{1}{2}} E^{-\frac{1}{4}},\ E^{\frac{3}{4}} \alpha^{\frac{1}{2}} (\sigma S)^{-\frac{3}{4}},\ (\sigma S)^{\frac{5}{4}} \alpha^{\frac{1}{2}} E^{-\frac{1}{4}}) \tag{6.12}$$

in order to satisfy condition (6.2 a). The solution for layer $B7c$ is given by (5.29), where $b^i(r_0)$ is found from (4.29). The solution for layer $B5$ is given by (5.20), where

$$v_n = (4\sigma S \alpha^2)^{\frac{1}{4}} E^{-\frac{1}{4}} 2(n\pi)^{-1} [1 + (-1)^n] b^i(r_0). \tag{6.13}$$

(8) If $E^{\frac{1}{2}} \alpha^{-1} \ll \sigma S \ll \alpha^6 E^{-1}$ and $\alpha^2 \ll 1$ the interior variables are given by (4.31), (4.33) and (4.29) and side-wall layer $B2b$ has scaling

$$(v, b) = O(\alpha^{-1}, E^{\frac{1}{2}} \alpha^{-2}). \tag{6.14}$$

The solution for layer $B2b$ is given by (5.12) and (6.10).

(9) If $\alpha^2 E \ll \sigma S E \ll \alpha^6$ and $(\sigma S)^2 \alpha^2 \ll E$ the interior variables are given by (4.31), (4.33) and (4.29) again and we have layers $B7c$ and $B5$ with respective scalings

$$(v, b) = O((\sigma S)^{\frac{1}{2}} \alpha^{-1},\ E^{\frac{1}{2}} \alpha^{-2}) \tag{6.15}$$

and

$$(v, \psi, b) = O((\sigma S)^{\frac{1}{2}} \alpha^{-1},\ E(\sigma S)^{-\frac{1}{2}} \alpha^{-1},\ (\sigma S)^{\frac{3}{2}} \alpha^{-1}). \tag{6.16}$$

The solution for layer $B7c$ is given by (5.29), where $b^i(r_0)$ is found from (4.29). The solution for layer $B5$ is given by (5.20) and (6.13).

(10) If $\sigma S \ll \alpha^2 \ll E(\sigma S)^{-2}$ and $E^{\frac{1}{2}} \ll \sigma S$ the interior variables are given by (4.31), (4.33) and (4.29) once more and we now have layers $B7b$ and $B5$ with respective scalings

$$(v, b) = O(E^0, E^{\frac{1}{2}} \alpha^{-2}) \tag{6.17}$$

and (5.17 b). The azimuthal velocity within the side-wall layers is no longer large as it was in cases (6)–(9); therefore the interior azimuthal velocity (4.33) again becomes important in the side-wall boundary condition (6.2 a). The solution for layer $B7b$ is given by (5.27), where $b^i(r_0)$ is found from (4.29). The solution for layer $B5$ is given by (5.20), where

$$v_n = 2(n\pi)^{-1} [(-1)^n v_B(r_0) - v_T(r_0)]. \tag{6.18}$$

(11) If $\sigma S E^{-1} \ll \alpha^2 \ll 1$ and $E^{\frac{2}{3}} \ll \sigma S \ll E^{\frac{1}{2}}$ the relevant interior variables are still given by (4.31), (4.33) and (4.29) but the side-wall layers now are $B5$, $B7a$ and $B8$. Layer $B8$ satisfies condition (6.2 a) to dominant order; that solution is given by (5.33) and (6.4). Layer $B7a$ satisfies condition (6.2 b); that solution is given by (5.25), where

$$b^i(r_0) = \tfrac{1}{4} E^{\frac{1}{2}} \alpha^{-2} [v_T(r_0) - v_B(r_0)]. \tag{6.19}$$

Layer $B5$ satisfies condition (6.2 a) to order $\sigma S E^{-\frac{1}{2}}$. In this parameter region azimuthal velocities of this order are generated within layers $B7a$ and $B8$ but the laminated flow in the interior is weaker than $O(\sigma S E^{-\frac{1}{2}})$. With scaling (5.17 a) the solution for layer $B5$ is given by (5.19) and (6.6).

† Because of the curious nature of layer $B7$ the order of magnitude of v is not continuous where parameter region (6) adjoins region (7). The order of magnitude of v would have been continuous there if $B7$ had retained its predicted scale. There is a similar disparity where region (8) adjoins regions (9) and (10).

(12) If $E\alpha^{-2} \ll \sigma S \ll E^{\frac{1}{2}}$ and $\alpha^2 \ll 1$ we must satisfy condition (5.4) and the relevant interior variables are given by (4.31), (4.33), (4.29) and (4.30). The side-wall layers are $B1$, $B7a$ and $B8$. Layers $B1$ and $B8$ play the same role as in case (1) and their solutions are given by (5.7), (5.33) and (6.3). Layer $B7a$ satisfies condition (5.4d) with (5.25) and (6.19).

This completes the analysis of the closure problem and assures us that the problem is well posed and the scaling is correct.

7. Summary and discussion

Certain slow, steady, mechanically driven, axisymmetric motions of a stably stratified, electrically conducting fluid have been studied. Attention has been focused upon the parameter values for which hydromagnetic effects first become important in a rotating stratified fluid and upon the nature of their influence on the flow of that fluid.

The possible horizontal and vertical length scales which may occur in the fluid are catalogued in tables 1–3 and in figures 1 and 2 and are discussed in appendix A. In addition to the known length scales which occur in rotating stratified flow and in rotating hydromagnetic flow, several new scales involving both buoyant and magnetic forces may occur. One pair of new scales, labelled $A7$ and $B7$ in appendix A, are of particular interest because they occur for relatively small values of stratification and magnetic field, specifically when $E \ll \sigma S\alpha^2$. In terms of dimensional parameters the condition for these new scales to occur is

$$\bar{\alpha}\Delta T g L\bar{\sigma}B^2/\rho\Omega^2\kappa \gg 1.$$

Note that this condition is independent of viscosity.

The most important part of the analysis is contained in §3, where the nature of the interior flow is determined. In this section the vertical shear of the azimuthal velocity (referred to as the laminated flow in L) is ordered with respect to the Ekman number:

$$\partial v/\partial z = O(E^{+k}).$$

A set of five constraints upon the size of k are found such that the mechanical forcing for the problem remains of dominant order. Two of these constraints arise from the azimuthal and meridional velocities in the interior [see (3.10) and (3.11)], two arise from the azimuthal and meridional velocities in the end-wall boundary layer $A7$ [see (3.12) and (3.13)] and one arises from the azimuthal velocity in the end-wall boundary layer labelled $A2b$ [see (3.14)]. This latter layer is essentially the Hartmann layer, which can occur on a scale much thicker than the Ekman layer under conditions of weak magnetic field and strong stratification (specifically if $E \ll \alpha^2 \ll E\sigma S$; see figure 2). Each constraint is dominant in a portion of the x, y plane, where the parameters x and y are defined by

$$\alpha^2 = E^{-x}, \quad \sigma S = E^{-y}.$$

The size of the exponent k is contoured on the x, y plane in figure 3.

Figure 3 reveals that the nature of the flow of a non-magnetic stratified rotating fluid, as described by Barcilon & Pedlosky (1967a, b), is altered by hydromagnetic effects if $E\alpha^{-2} \ll \sigma S \ll 1$ or if $E \ll \alpha^2$ and $1 \ll \sigma S$. The hydromagnetic interaction acts to *decrease* the shear in the azimuthal flow from the levels which would occur if $\alpha^2 = 0$. For relatively weak stratification ($\sigma S = O(E^{\frac{1}{2}})$) hydromagnetic effects become important in a confined stratified fluid if $\alpha^2 \geqslant O(E^{\frac{1}{2}})$. This is in agreement with the principal result of the similarity analysis of an unbounded fluid given in L. Further, if $\sigma S \geqslant O(1)$ hydromagnetic effects are important if $\alpha^2 \geqslant O(E)$. This transition occurs at a surprisingly small magnetic field strength, much weaker than that predicted in L.

It should be remarked that these very interesting results occur for a constant-heat-flux boundary condition. It was found in L that the transition to hydromagnetic flow was very sensitive to the thermal boundary condition prescribed. The nature of confined flow subject to a mixed (or constant temperature) thermal boundary condition remains to be determined.

Detailed solutions for the flow in the interior and in the end-wall boundary layers are presented in §4. The solutions for the side-wall boundary layers, which allow closure of the problem (i.e. satisfaction of the side-wall boundary conditions), are presented in §§5 and 6, verifying that the analysis is consistent. The nature of the new side-wall layer $B7$ is of particular interest because its scale is determined by the boundary conditions at the end walls in a manner similar to that for the $E^{\frac{1}{4}}$ side-wall layer. The actual solution for this layer is rather complicated since it involves analysis of the corner regions above and below the layer. It turns out that layer $B7$ exists on one of three different scales depending upon the values of the parameters.

The analysis in this paper has been performed for unit-order aspect ratio: $r_0 = O(1)$. As the aspect ratio becomes large, end-wall scale $A7$ increases (see table 2 with $l = r_0$) and the end-wall scale $A2b$, which results from a balance between [d] and [e] in (A 5), has a smaller region of validity. Therefore, both scales $A7$ and $A2b$, which play important roles in the dynamics of a confined fluid, are absent in the limit $r_0 \rightarrow \infty$.

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Appendix A

The purpose of this appendix is to survey the possible length scales which occur when various terms in (2.2)–(2.5) balance each other. First we shall obtain two scale equations; one for vertical scales (labelled case A) and one for horizontal scales (labelled case B). Following this we shall tabulate all possible scales resulting from these two scale equations. Finally we shall determine the regions in parameter space for which these scales may exist. We appear to have a three-parameter problem, involving E, σS and α^2, but since E is not an internal parameter, the parameter space is in fact a parameter plane. It is convenient to introduce

$$\alpha^2 = E^{-x}, \quad \sigma S = E^{-y} \tag{A 1}$$

and to study the x, y plane. This eliminates E from direct consideration and allows definite numerical values to be assigned on the x and y axes. The signs in the exponents in (A 1) have been chosen such that $d(\alpha^2)/dx$ and $d(\sigma S)/dy$ are positive when $E \ll 1$.

The scale analysis is based upon the operator for a single variable. If we eliminate T, ψ and b from (2.2)–(2.5) in favour of v we obtain

$$E(\nabla^2 - r^{-2})\,\mathscr{L}^2(v) + 4E(\nabla^2 - r^{-2})^2 v_{zz} + \sigma S \mathscr{L}[(rv)_r/r]_r = 0, \qquad \text{(A 2)}$$

where $$\mathscr{L} \equiv E(\nabla^2 - r^{-2})^2 - 2\alpha^2\,\partial^2/\partial z^2.$$

Note that if we assume that the azimuthal velocity v is linear in radius r, as it is in von Kármán similarity, then the third term of (A 2) is identically zero and the stratification no longer plays a role in the scale analysis.

In order to abstract the scale structure from (A 2), we shall neglect curvature terms and approximate derivatives by

$$\partial/\partial z = h^{-1}, \quad \partial/\partial r = l^{-1}, \qquad \text{(A 3)}$$

where h and l represent the vertical and horizontal length scale respectively. (This notation follows that of Blumsack 1972).

In horizontal boundary layers or shear zones, the horizontal scale of motion is much greater than the vertical scale:

$$h \ll l. \qquad \text{(A 4)}$$

With (A 3) and (A 4), (A 2) may be converted to

$$\underset{[a]}{\frac{E^3}{h^8}} + \underset{[b]}{\frac{E\alpha^4}{h^4}} + \underset{[c]}{\frac{E}{h^4}} + \underset{[d]}{\frac{E\sigma S}{h^2 l^2}} + \underset{[e]}{\frac{\sigma S \alpha^2}{l^2}} = 0 \quad (\text{case } A). \qquad \text{(A 5)}$$

The terms in (A 5) have been labelled to allow individual reference. In this abstraction signs, numerical coefficients and the magnitude of the dependent variable are irrelevant. Equation (A 5) will be used to determine the possible vertical scales h. Similarly, with (A 3) and

$$l \ll h \qquad \text{(A 6)}$$

the possible horizontal scales l may be determined from

$$\underset{[a]}{\frac{E^3}{l^8}} + \underset{[b]}{\frac{E\alpha^4}{h^4}} + \underset{[c]}{\frac{E}{h^2 l^2}} + \underset{[d]}{\frac{E\sigma S}{l^4}} + \underset{[e]}{\frac{\sigma S \alpha^2}{h^2}} = 0 \quad (\text{case } B). \qquad \text{(A 7)}$$

The terms in (A 7) have been labelled for individual reference in a manner identical to that in (A 5) because the corresponding terms in (A 5) and (A 7) originate from the same terms in (A 2) and represent the same physical effect.

The full equation (A 2) is of tenth order in both z and r and the full set of boundary conditions (2.7) consist of two conditions on ψ and one condition on each of v, T and b at each boundary. The abstractions (A 5) and (A 7) represent equations which are of only eighth order in z and r. Therefore not all the boundary conditions can be satisfied locally near the boundaries. This permits the interior

	$\alpha \neq 0$ $\sigma S \neq 0$	$\alpha \neq 0$ $\sigma S = 0$	$\alpha = 0$ $\sigma S \neq 0$	$\alpha = 0$ $\sigma S = 0$
Full equation (A 2)	10	8	6	6
Case A : $\partial/\partial z$ only (A 5)	8	4	6	4
Case B : $\partial/\partial r$ only (A 7)	8	8	6	6

TABLE 5. Orders of the terms in the governing equations for various parameter ranges

flow to be determined. Further reductions in order occur if σS or α^2 or both are set equal to zero. The respective orders for each possibility are presented in table 5. Each reduction in order corresponds to modes which do not possess intrinsic length scales but exist on scales determined by the boundary conditions. For example if $v_z = \psi_{zz} = 0$ terms [5] and [7] in (2.3) may balance for a wide range of horizontal length scales satisfying conditions (A 6). The appropriate scale is then determined by the boundary conditions. In homogeneous non-magnetic fluids this leads to the $E^{\frac{1}{4}}$ side-wall scale. Recently Vempaty & Loper (1975) have found that this mode occurs in homogeneous hydromagnetic fluids with a horizontal scale $(E/\alpha^2)^{\frac{1}{4}}$ for $1 \ll \alpha^2 \ll E^{-\frac{1}{3}}$. (Ingham (1969) predicts the wrong scale for this mode.) The scale analysis below is incapable of predicting this important mode but it has been included in the side-wall analysis of § 5. The scale analysis yields a limited set of possible length scales which excludes all scales resulting from time dependence, non-axisymmetry, nonlinearity (such as those found by Barcilon 1970; Loper 1972), Froude number effects (such as that found by Loper 1975), anisotropic diffusivity or beta-plane effects (as discussed by Blumsack 1973).

The scale analysis proceeds as follows. The possible vertical or horizontal length scales may be determined by systematically equating two terms of (A 5) or (A 7) respectively. Each possible scale is numbered and its properties are recorded in tables 1–3. In order that the chosen terms be dominant the remaining three terms of (A 5) or (A 7) must be small. In addition condition (A 4) or (A 6) must be satisfied. These existence conditions place limits on the range of the parameters E, σS and α^2 (or, more simply, x and y) for which the chosen scale may exist. This information is compiled in figures 1 and 2. We now turn to a brief discussion of the possible length scales and force balances then complete this appendix with a discussion of figures 1 and 2.

In homogeneous non-magnetic flow ($\sigma S = \alpha^2 = 0$), (A 5) and (A 7) each reduce to two terms, [a] and [c], yielding a single scale labelled $A1$ or $B1$. Scale $A1$ is the well-known Ekman vertical scale and scale $B1$ is the Stewartson $E^{\frac{1}{3}}$ scale. In each scale the dominant force balance is between Coriolis and viscous forces (see table 1).

In homogeneous hydromagnetic flow ($\sigma S = 0, \alpha^2 \neq 0$), (A 5) and (A 7) each contain three terms: [a], [b] and [c]. This appears to allow two hydromagnetic length scales for each case, labelled $A2a$ or $B2a$ when [a] and [b] balance and $A3$ or $B3$ when [b] and [c] balance. However it may be seen from (A 5) that a balance between [b] and [c] does not yield a scale for case A so scale $A3$ does not exist.

Scale $A2a$ is the well-known Hartmann scale and scale $B2a$ is the non-rotating hydromagnetic side-wall scale (see Roberts 1967, pp. 186–190). These scales result from a balance between magnetic and viscous forces. Scale $B3$ is a hydromagnetic side-wall scale recently investigated by Vempaty & Loper (1975) and results from a balance between Coriolis and magnetic forces.

In stratified non-magnetic flow ($\sigma S \neq 0$, $\alpha^2 = 0$), (A 5) and (A 7) again each contain three terms: [*a*], [*c*] and [*d*]. This allows two stratified length scales for each case, labelled $A4$ or $B4$ when [*a*] and [*d*] balance and $A5$ or $B5$ when [*c*] and [*d*] balance. Scale $A4$ is a buoyant Ekman scale while scale $B4$ is the buoyancy scale studied by Barcilon & Pedlosky (1967*b*). These scales result from a balance between viscous and buoyant forces. Scale $A5$ appears not to have been discussed in the literature although it occurs in the solution of example 2 in Barcilon & Pedlosky (1967*a*). Scale $B5$ is the hydrostatic scale discussed by Barcilon & Pedlosky (1967*b*). These two scales result from a balance between Coriolis, viscous and buoyant forces. Scales $B4$ and $B5$ correspond to the scales *c* and *e* discussed by Blumsack (1973).

In stratified hydromagnetic flow ($\sigma S \neq 0$, $\alpha^2 \neq 0$), (A 5) and (A 7) each contain all five terms. This would seem to imply that we should find five additional scales for each case which involve both buoyant and magnetic forces. However the number of additional scales may be reduced by closer inspection of (A 5) and (A 7). In each case if term [*a*] is greater than term [*b*] then term [*d*] is greater than term [*e*] and vice versa. This means that [*a*] and [*e*] can never balance and that [*b*] and [*d*] can never balance. The three pairs of new scales which remain are labelled $A2b$ or $B2b$ when [*d*] and [*e*] balance, $A6$ or $B6$ when [*b*] and [*e*] balance and $A7$ or $B7$ when [*c*] and [*e*] balance. Scales $A2b$ and $B2b$ are identical in size to $A2a$ and $B2a$ (hence the labelling) but the force balances and regions of existence differ (see table 1 and figure 2). Scale $A6$ is a new scale resulting from a balance between buoyant and magnetic forces; since [*b*] and [*e*] in (A 7) fail to contain l, scale $B6$ does not exist. Scales $A7$ and $B7$ are also new, resulting from a balance between Coriolis, buoyant and magnetic forces. Scales $A7$ and $B7$ are of particular interest because they can occur for relatively weak stratification and magnetic fields (see figure 2). It is seen in § 3 that scales $A2b$ and $A7$ are instrumental in determining the nature of the interior flow over a wide parameter range.

The properties of the possible length scales discussed above are presented in tables 1–3. In presenting the length scales for each case in tables 2 and 3, allowance is made for the larger scale (l in case A or h in case B) not to be of unit order. This is necessary because a given force balance may occur simultaneously on two separate scales. For example, Vempaty & Loper (1975) have found that scale $B3$ occurs simultaneously with scales $h = 1$, $l = \alpha^{-2}$ and $h = (E\alpha^6)^{\frac{1}{4}}$, $l = (E/\alpha^2)^{\frac{1}{4}}$. This completes the discussion of the possible length scales. We now turn to a discussion of the parameter range for which each scale may exist.

Each of the scales discussed above results from a particular balance between forces due to viscosity, rotation, stratification and magnetic fields. Viscous forces are important on some scale for all values of σS and α^2 [or of x and y; see (A 1)]. However the remaining three types of forces each are important on some scale only for a limited range of parameters. The regions in which magnetic (M), buoyant (A for Archimedean) or Coriolis (C) forces may be important are drawn on the x, y plane in figure 1. Seven regions occur, in which M, A and C are important singly, in pairs or all three simultaneously. For parameter values in the lower left region only Coriolis forces may balance viscous forces and the flow is homogeneous and non-magnetic to dominant order. Flow in this parameter region has received a great deal of attention in the literature. For parameter values in the lower right region magnetic forces balance viscous and flow is homogeneous and non-rotating to dominant order while at the upper left, buoyant forces are dominant and the flow is essentially non-rotating and non-magnetic. Barcilon & Pedlosky (1967a, b) have investigated the nature of the flow which occurs for parameter values in the regions marked (C) and (A & C) in figure 1. The nature of the flow which occurs for parameter values in the regions marked (C), (M & C) and M has been investigated by Ingham (1969) and Vempaty & Loper (1975).

The regions of the parameter plane for which the various length scales presented in tables 2 and 3 may occur are drawn in figure 2. This figure is more complicated than figure 1: a total of fourteen separate regions now occur. This fragmentation of the parameter regions occurs because more than one combination of scales may exist in a given force-balance region. For example, in the region where M, A and C are all important, a total of six different scale combinations may exist. Much of this fragmentation is due to the horizontal or side-wall scales (case B). Fortunately the interior flow is controlled by the vertical or end-wall scales (case A) for a wide parameter range. Therefore the nature of the interior flow may be determined in § 3 more simply than figure 2 appears to indicate.

Appendix B

The purpose of this appendix is to derive a differential equation governing the variation of the variable b within layer $B7$ which does not have the degeneracy of (5.22) when $b_z = 0$.

Let us begin by assuming that v_z, ψ and b are independent of z in layer $B7$. Further, owing to symmetry, assume that v is odd about $z = \frac{1}{2}$. This allows us to write

$$v(\delta, z) = w(\delta)(1-2z), \tag{B 1}$$

where $w(\delta) = v(\delta, 0)$. Now (5.21 a, c) become

$$\psi = (4E/\sigma S)\,w, \quad -2w + (\sigma S\alpha^2/8E)\,b_{\delta\delta} = 0. \tag{B 2}$$

Equation (5.21 b) is useless and we need to obtain an additional relationship between ψ, b and w by considering the boundary conditions.

The Ekman compatability conditions beneath layer $B7$ are homogeneous [v_B and v_T do not vary on the scale of δ] and the forcing is provided by the condition on b at the side wall. Using (2.10) and (B 1) the conditions are

$$b + b^* = \tfrac{1}{2}E^{\frac{1}{2}}[w \mp v^*], \quad \psi + \psi^* = \tfrac{1}{2}E^{\frac{1}{2}}[w \pm v^*], \quad T_z^* = 0 \quad \text{at} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}, \tag{B 3a–c}$$

$$b + b^i = 0, \quad b^* = T_\delta^* = 0 \quad \text{at} \quad r = r_0, \quad \delta = 0, \tag{B 3d, e}$$

where an asterisk denotes an *unscaled* corner-region variable.

In order to relate b and w through the boundary conditions, it is necessary to consider the corner regions; by symmetry it is sufficient to consider only that near $z = 0$. The equations governing the corner regions include the dynamics of layer $A7$ and it is necessary to re-introduce the thermal perturbation T. Using table 2, we see that the equations governing the lower corner region are

$$2v^*_\zeta = -T^*_\delta, \quad \psi^*_\zeta = \alpha^2 b^*_\zeta, \qquad \text{(B 4}\,a, b)$$

$$\psi^*_\delta = (E\alpha^2/8\sigma S)^{\frac{1}{2}}(T^*_{\zeta\zeta}+T^*_{\delta\delta}), \qquad \text{(B 4}\,c)$$

$$v^*_\zeta + (\sigma S\alpha^2/8E)^{\frac{1}{2}}(b^*_{\zeta\zeta}+b^*_{\delta\delta}) = 0, \qquad \text{(B 4}\,d)$$

where $\zeta = (\sigma S\alpha^2/8E)^{\frac{1}{2}}z$.

The difficulty with layer $B7$ stems from the fact that (5.21 b) may be integrated to $\psi = \alpha^2 b + f(r)$, where $f(r)$ is an arbitrary function. A similar integration of (B 4 b) leads to

$$\psi^* = \alpha^2 b^*, \qquad \text{(B 5)}$$

where the function of integration has been set equal to zero using the condition that ψ^* and b^* tend to zero as we move out of the corner region in the axial direction.

In the parameter region under consideration the layer $B7$ is thicker than $O(E^{\frac{1}{2}})$. It follows from (5.21 c) and (B 4 d) that $E^{\frac{1}{2}}v \ll b$ and $E^{\frac{1}{2}}v^* \ll b^*$. This allows the condition at $z = \zeta = 0$ to be simplified to

$$b+b^* = 0, \quad T^*_\zeta = 0, \quad \psi - \alpha^2 b = -\tfrac{1}{2}E^{\frac{1}{2}}(w+v^*) \quad \text{at} \quad \zeta = 0. \qquad \text{(B 6}\,a\text{–}c)$$

Condition (B 6 c) will provide the relationship needed between ψ, b and w once we know v^* in terms of those variables.

If we extend $b(\delta, 0)$ as an *odd* function of δ about $\delta = 0$ and solve (B 4) and (B 5) on the domain $-\infty < \delta < \infty$, $0 < \zeta < \infty$, the side-wall conditions on b^* and T^* [see (B 3)] are automatically satisfied and we can introduce the complex Fourier transform to solve the corner-region equations:

$$\bar{f}(p,\zeta) = \int_{-\infty}^{\infty} f(\delta,\zeta)\exp(ip\delta)\,d\delta,$$

$$f(\delta,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(p,\zeta)\exp(-ip\delta)\,dp.$$

The transformed equations are

$$2\bar{v}^*_\zeta = ip\bar{T}^*, \quad \bar{\psi}^* = \alpha^2\bar{b}^*, \qquad \text{(B 7}\,a, b)$$

$$-ip\bar{\psi}^* = (E\alpha^2/8\sigma S)^{\frac{1}{2}}[\bar{T}^*_{\zeta\zeta} - p^2\bar{T}^*], \qquad \text{(B 7}\,c)$$

$$\bar{v}^*_\zeta + (\sigma S\alpha^2/8E)^{\frac{1}{2}}[\bar{b}^*_{\zeta\zeta} - p^2\bar{b}^*] = 0, \qquad \text{(B 7}\,d)$$

with conditions

$$\bar{b}+\bar{b}^* = 0, \quad \bar{T}^*_\zeta = 0 \quad \text{at} \quad \zeta = 0. \qquad \text{(B 8)}$$

The solutions of (B 7) which satisfy conditions (B 8) are

$$\bar{b}^* = -\frac{\bar{b}}{\lambda_1+\lambda_2}[\lambda_2 \exp(-\lambda_1\zeta)+\lambda_1 \exp(-\lambda_2\zeta)], \tag{B 9a}$$

$$\bar{T}^* = \left(\frac{2\sigma S\alpha^2}{E}\right)^{\frac{1}{2}} \frac{\bar{b}}{\lambda_1+\lambda_2}[\lambda_2 \exp(-\lambda_1\zeta)-\lambda_1 \exp(-\lambda_2\zeta)], \tag{B 9b}$$

$$\bar{v}^* = \left(\frac{\sigma S\alpha^2}{2E}\right)^{\frac{1}{2}} \frac{ip\bar{b}}{\lambda_1+\lambda_2}\left[-\frac{\lambda_2}{\lambda_1}\exp(-\lambda_1\zeta)+\frac{\lambda_1}{\lambda_2}\exp(-\lambda_2\zeta)\right], \tag{B 9c}$$

where $\lambda_1^2 = p^2+2ip$ and $\lambda_2^2 = p^2-2ip$.

Evaluating (B 9c) at $\zeta = 0$ provides a relation between $\bar{v}^*$ and $\bar{b}$:

$$\bar{v}^* = \left(\frac{\sigma S\alpha^2}{2E}\right)^{\frac{1}{2}} \left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right) ip\bar{b}. \tag{B 10}$$

This may be inverted using the convolution theorem to read

$$v^*(\delta, 0) = \left(\frac{\sigma S\alpha^2}{2E}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} t(\eta)\, b(\delta-\eta)\, d\eta, \tag{B 11}$$

where
$$t(\delta) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \left[\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right] p \exp(-ip\delta)\, dp. \tag{B 12}$$

The solution for the variables in the corner region have negative exponential dependence on ζ. It follows from this fact and from (B 4d) that $\operatorname{sgn} v^* = \operatorname{sgn} b^*$. Next, using (B 6a) we have $\operatorname{sgn} v^* = -\operatorname{sgn} b$. From this we can conclude that the sign of $t(\delta)$ must be *negative*. Also noting that b has been defined as odd about $\delta = 0$ and noting from (B 4) that v^* is odd as well, we see that $t(\delta)$ must be *even* about $\delta = 0$.

Equation (B 11) may be combined with (B 6c) to provide an equation relating ψ, b and w. This equation, plus (B 2), yields a set of three equations for these three unknowns. Before combining (B 11) with (B 6c) let us first evaluate $t(\delta)$. Rather than calculating $t(\delta)$ using (B 12), it is easier to evaluate

$$u(\delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right] \exp(-ip\delta)\, dp \tag{B 13}$$

and then find $t(\delta)$ from

$$t(\delta) = du/d\delta. \tag{B 14}$$

In light of the discussion of t it follows that $u(\delta)$ is *odd* about $\delta = 0$ and that $u(\delta)$ is *positive* for $\delta > 0$. By symmetry considerations we need only find $u(\delta)$ for $\delta > 0$. In this case the integral (B 13) may be evaluated by closing the contour in the lower half-plane and contributions to u come only from λ_1. If we let $p = iq$ then (B 13) becomes

$$u(\delta) = \frac{1}{2\pi}\int_{-i\infty}^{i\infty} q^{-\frac{1}{2}}(q+2)^{-\frac{1}{2}} \exp(\delta q)\, dq,$$

which has the form of a Laplace inversion. From Erdélyi (1954, p. 235, §5) we have

$$u(\delta) = e^{-\delta} I_0(\delta), \tag{B 15}$$

where I_0 is a modified Bessel function. From (B 14) we have

$$t(\delta) = e^{-\delta}[I_0(\delta) - I_1(\delta)]. \tag{B 16}$$

Equation (B 16) is valid only for $\delta > 0$. The solution for $\delta < 0$ follows from the fact that $t(\delta)$ is even about $\delta = 0$.

We may now combine (B 2), (B 6*c*) and (B 11) into a single integro-differential equation for b:

$$(1 + \sigma S/8E^{\frac{1}{2}})\, b_{\delta\delta} - 4b + (2\sigma S/\alpha^2)^{\frac{1}{2}} \int_{-\infty}^{\infty} t(\eta)\, b(\delta - \eta)\, d\eta = 0. \tag{B 17}$$

This equation may be reduced to a differential equation by the following argument. From (B 16) we see that $t(\delta)$ is of order unity and varies with δ on a scale $\delta = O(1)$. The integral term in (B 17) is important only if $\alpha^2 \ll \sigma S$ $(x < y)$. Since layer $B7$ exists only if $E \ll \sigma S \alpha^2$ $(-1 - x < y)$ it follows that $E^{\frac{1}{2}} \ll \sigma S$ $(-\frac{1}{2} < y)$ whenever $\alpha^2 \ll \sigma S$ (see figure 4). Therefore whenever the integral term is important (B 17) reduces to

$$(\sigma S \alpha^2/2E)^{\frac{1}{2}}\, b_{\delta\delta} + 8 \int_{-\infty}^{\infty} t(\eta)\, b(\delta - \eta)\, d\eta = 0.$$

In this case the scale of variation of b is greater than that predicted by the scale analysis in appendix A. If b varies with δ on a scale greater than $\delta = O(1)$, we may approximate the integral by

$$\int_{-\infty}^{\infty} t(\eta)\, b(\delta - \eta)\, d\eta = b(\delta) \int_{-\infty}^{\infty} t(\eta)\, d\eta.$$

From (B 14) and (B 15) it is easy to see that

$$\int_{-\infty}^{\infty} t(\eta)\, d\eta = 2 \int_{0}^{\infty} t(\eta)\, d\eta = 2(u(\infty) - u(0)) = -2.$$

We finally obtain from (B 17) the differential equation governing the variation of b in layer $B7$:

$$(1 + \sigma S/8E^{\frac{1}{2}})\, b_{\delta\delta} = [4 + 2(2\sigma S/\alpha^2)^{\frac{1}{2}}]\, b. \tag{B 18}$$

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